# HW 3 Solutions



**1 a** Let ~~a~~ us ~~see~~ use the fact that $\mathbb{N} \times \mathbb{N}$ is countable. Given a denumerable collection of denumerable sets, Write the sets as $\{A_i : i \in \mathbb{N}\}$.

Each $A_i = \{a_{i1}, a_{i2}\ a_{i3}, \ldots\}$.

Now ~~a~~ Consider the bijection given by $(i,j) \longleftrightarrow a_{ij}$, $(i,j) \in \mathbb{N} \times \mathbb{N}$

This shows that $\cup A_i$ is denumerable.

**1 b** Let $p_1, p_2, \ldots, p_d$ be the first $d$ prime numbers. Consider the map from $\mathbb{N}^d$ into $\mathbb{N}$ given by

$$(x_1, \ldots, x_d) \longrightarrow p_1^{x_1} \cdots p_d^{x_d}.$$

This is 1-1 by uniqueness of prime factorization! (This map is into, as no ~~numb~~ pt in $\mathbb{N}^d$ is mapped onto

a prime greater than $p_d$.) This 
shows $\mathbb{N}^d$ is denumerable. You can also use part (a) to do this problem.

Induct on dimension $d$: $d=1$ is obviously.
Assuming $\mathbb{N}^d$ is denumerable, ~~it~~ write

$$\mathbb{N}^{d+1} = \{1\} \times \mathbb{N}^d \cup \{2\} \cup \mathbb{N}^d \cup \{3\} \cup \mathbb{N}^d \cup \cdots$$

That is $\mathbb{N}^{d+1}$ is a denumerable union of copies of $\mathbb{N}^d$.

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The union of convex need not be convex. Intervals are convex, but $(-1, 0) \cup (1, 2)$ ~~need~~ is certainly not convex.

If $K_1$, $K_2$ are convex, then $K = K_1 \cap K_2$ is convex. Given $x, y \in K$, then $\{\alpha x + (1-\alpha) y \mid 0 \le \alpha \le 1\} \subset K_1$ & is also contained in $K_2$. So the proof is done.

**90.** Suppose $A, B$ are ~~open~~ intervals.

Then $A, B$ are open in $\mathbb{R}$ iff $A \times B$ is open in $\mathbb{R}^2$. This is easy to check.

Now an open set $A$ in $\mathbb{R}$ is a union of open intervals $A = \bigcup_{x \in A} I_x$. So if $B = \bigcup_{y \in B} J_y$ is also open then $A \times B = \bigcup_{\substack{x \in A \\ y \in B}} I_x \times J_y$

so $A \times B$ is the union of open sets, hence open.

Conversely suppose $A \times B$ is not open so we may choose $(x, y) \in A \times B$, ~~and open~~ so that ~~$I \times J$ a ne for all~~ ~~neighborhoods open~~ ~~$I \times J$ of $(x,y)$.~~ ~~$I \times J \notin A \times B$.~~ ~~for all n~~ ~~$(x - \frac{1}{n}, x + \frac{1}{n}) \times (y - \frac{1}{n}, y + \frac{1}{n})$~~ ~~of $(x$~~ for all $n$, $(x - \frac{1}{n}, x + \frac{1}{n}) \times (y - \frac{1}{n}, y + \frac{1}{n})$ is not contained in $A \times B$. That must mean that e.g. $(x - \frac{1}{n}, x + \frac{1}{n}) \not\subset A$. But then ~~x~~ $A$ is not open.

9 G : ~~Here is the point~~ Certainly if A is a union of a countable no. of open sets, then A is open. Conversely, we note the following: ~~Assistance~~ Let $\mathcal{R}$ be the collection of open sets $R = R_1 \times \cdots \times R_d$ where each $R_j$ is an open interval with Rational end points. $\mathcal{R}$ is countable.

Moreover, suppose A is open and $x \in A$. Then for some $\varepsilon > 0$, A contains the ball of radius $\varepsilon$ centered at $x$. But this ball also contains a rectangle $R \in \mathcal{R}$ with $x \in R$. The remainder of the argument is easy.

4. Clearly $\mathbb{Z}^d$ is countable. And we can place $\mathbb{Z}^d$ in 1-1 correspondence with polynomials of degree d. Each such polynomial has at most d roots. So the algebraic numbers which arise from d-dim'l polynomials is countable. The countable union of countable collections is countable, so the set of algebraic integers is countable.