Analysis I: Second Exam
February 27, 2006

Complete FIVE of the SIX problems below.
Complete Each Question on One Sheet of Paper, with Name on Each Sheet
Front Side of the Paper Only!
Give Complete and Clear Solutions! All problems are worth 20 points.

1. Prove *from the definition of compactness* that the two sets below are compact.

   (a)
$$A = \{0\} \cup \{2^{-n} : n = 1, 2, \ldots\} = \{\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots, 0\}$$

   (b)
$$B = \{0\} \cup \{2^{-n} : n = 0, 1, 2, \ldots\} \cup \{2^{-n} + 2^{-m} : m, n = 1, 2, \ldots, m > n\}.$$

Solution:

(a) Let $\mathcal{O}$ be an open cover of A. One element U_1 must contain 0. U_1 is open, so it contains an interval $(-2^{-n_0}, 2^{-n_0})$. So U_1 contains not only 0 but also all points 2^{-n} for $n > n_0$. It remains to cover the points $\frac{1}{2}, \frac{1}{4}, \ldots, 2^{-n_0}$. Clearly, there are at most n_0 open sets in $\mathcal{O}$ which will do this, so we are finished.

(b) Let us remark that the finite union of compact sets is compact, and that this is very easy to prove. Let us also remark that for all n the set
$$B_n = \{2^{-n}\} \cup \{2^{-n} + 2^{-m} : m = n+1, n+2, \ldots\}$$
is compact, as follows from the first part of the problem. Now, let us consider the set in question. Let $\mathcal{O}$ be an open cover of B. One element U_1 must contain 0. U_1 is open, so it contains an interval $(-2^{-n_0}, 2^{-n_0})$. So U_1 contains not only 0 but also all points 2^{-n} and $2^{-n} + 2^{-m}$ for $m > n > n_0$. It remains to find the finite subcover for $B_1 \cup \cdots \cup B_{n_0}$. But this is a finite union of compact sets, and $\mathcal{O}$ is a cover for this set, so there is a finite subcover $\mathcal{O}' \subset \mathcal{O}$ for this set. Then $\{U_1\} \cup \mathcal{O}'$ is a finite subcover for B.

2. Let A be a **compact** subset of $\mathbb{R}^d$. *Prove the following two assertions from the definition of compactness.*

   (a) Show that A is bounded.

   (b) If x is a cluster point of A then $x \in A$.

Solution:

Since we have to use the definition of compactness, this means that we will need to construct appropriate covers of A.

(a) Let $\mathcal{O}$ the the collection of open balls centered at the origin, with radius $n \in \mathbb{N}$. Since $\mathcal{O}$ covers $\mathbb{R}^d$, it certainly covers A. Hence there is a finite subcover $\mathcal{O}'$ of A. But, since the balls have increasing radius, this means that A must be contained in some ball of finite radius. That is, it is bounded.

(b) Seek contradiction. Assume that $x \notin A$. Let $\mathcal{O}$ be the open sets of the *complement* of the closed balls centered at x of radius 2^{-n} for $n \in \mathbb{N}$. Since A is compact, there is a finite subcover $\mathcal{O}'$ of A. But the sets in the cover are increasing. So there is a finite n so that $|x-y| > 2^{-n}$ for all $y \in A$. That is, x cannot be a cluster point of A.

3. (a) Show that if $A, B \subset \mathbb{R}$ are two open non–empty subsets of $(0,1)$, with $A \cup B = (0,1)$, then $A \cap B$ is non empty.

   (b) Show, from the defintion, that $(0,1)$ is connected.

Solution:

(a) Since A is non empty, there is a point $x \in A$. Let (a,b) be the longest interval which contains x and is contained in A. This is done by setting

$$a = \inf\{y \,:\, (y,x) \subset A\}, \qquad b = \sup\{z \,:\, (x,z) \subset A\}.$$

If $(a,b) = (0,1)$ we are done. If e.g. $0 < a < 1$ then $a \in B$. The set B is open, hence it contains a neighborhood of a, hence it contains an element of A.

(b) This is immediate from the first part. If $(0,1)$ were disconntected, we could find non empty disjoint open sets $A, B \subset (0,1)$ whose union is $(0,1)$. But this possibility was just removed by the first part of the problem.

4. Prove the Cantor Intersection Theorem: Let $F_1 \supset F_2 \supset \cdots$ a a sequence of decreasing, non empty, compact subsets of $\mathbb{R}^d$. Then $\bigcap F_k$ is not empty.

Solution:

There are two possible solutions.

(a) Supppose, by way of contradiction, that the intersection is empty. Then, consider the open sets $\{\mathbb{R}^d - F_k \,:\, k \geq 1\}$, which will then form an open cover of F_1. By compactness, there is there is a finite subcover, but the open sets are increasing, and so there is an open set $\mathbb{R} - F_K$ that contains F_1. But $F_K \subset F_1$, so we conclude that F_K must be empty, a contradiction.

(b) Choose, arbitrarily, points $x_k \in F_k$. If this sequence takes only finitely many values, $\{y_1, \ldots, y_n\}$, then clearly each of these values that occurs infinitely often is in the intersection $\bigcap F_k$. Otherwise, we have an infinite subset of a closed bounded set, hence it is has a cluster point y. y is a cluster point of each F_k, so $y \in F_k$, as the set is closed. Therefore, $y \in \bigcap F_k$.

5. Prove this instance of the Balzono Wierstrauss Theorem: Every infinite bounded set $A \subset [0,1]$ has a cluster point. (You should of course only use the fundamental properties of the real line, such as the supremum property, or the nested cell property.)

Solution:

(a) Construct a sequence of decreasing dyadic intervals I_j, $j \geq 0$, by setting $I_0 = [0,1]$. In the inductive stage, given I_j, choose I_{j+1} either the left or right half of I_j, subject to the condition that $I_{j+1} \cap A$ has infinite cardinality. Then, the assertation is that the unique point $x \in \bigcap I_j$ is a cluster point of A. Indeed, each neighborhood of x must contain some I_j, hence the neighborhood has infinitely many points of A in it.

(b) Let $S = \{x \,:\, [0,x] \cap A \text{ has finite cardinality}\}$. This is a bounded set, hence has a supremum σ. Arguing much as above, one can see that σ is a cluster point of A.

6. Show that the set below is a connected subset of $\mathbb{R}^2$.

$$\{(0,y) \,:\, -1 \leq y \leq 1\} \cup \{(x, \sin(1/x)) \,:\, 0 < x < 1\}.$$

Solution: Suppose it is disconnected by open sets A and B. Certainly the vertical line segment $\{(0,y) \,:\, -1 \leq y \leq 1\}$ is connected, and so must lie entirely in e.g. A. Likewise, for all $0 < \epsilon < 1$, the set

$$G_\epsilon = \{(x, \sin 1/x) \,:\, \epsilon < x < 1\}$$

is an image of the unit interval, hence connected, and so, must be entirely inside of either A or B. Now, A is open, hence it contains a neighborhood of the vertical line segment. Therefore, for some $\epsilon > 0$ we must have $A \cap G_\epsilon \neq \emptyset$. This precludes G_ϵ from being contained in B, and so establishes a contradiction.