This week’s quiz: covers *Sections 5.1 and 5.2*

Midterm 3, on November 17th (next Friday)

- Exam covers: Sections 3.1, 3.2, 5.1, 5.2, 5.3 and 5.5
Section 5.3

Diagonalization
Many real-word (linear algebra problems):

- Start with a *given situation* \((v_0)\) and
- want to know *what happens after some time* (iterate a transformation):

\[
v_n = A v_{n-1} = \ldots = A^n v_0.
\]

- Ultimate question: *what happens in the long run* (find \(v_n\) as \(n \to \infty\))

Old Example

Recall our example about *rabbit populations*: using eigenvectors was easier than matrix multiplications, but …

- Taking *powers of diagonal* matrices is easy!
- Working with *diagonalizable matrices* is also easy.
Powers of Diagonal Matrices

If $D$ is diagonal

Then $D^n$ is also diagonal, the diagonal entries of $D^n$ are the \textit{nth powers of the diagonal} entries of $D$.
Powers of Matrices that are Similar to Diagonal Ones

When is $A$ not diagonal?

**Example**

Let $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$. Compute $A^n$. Using that $A = PDP^{-1}$ where $P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

From the first expression:

\[
A^2 = \\
A^3 = \\
\vdots \\
A^n = \\
\]

Plug in $P$ and $D$:

\[
A^n = 
\]
Diagonalizable Matrices

**Definition**
An \( n \times n \) matrix \( A \) is **diagonalizable** if it is similar to a diagonal matrix:

\[
A = PDP^{-1} \quad \text{for } D \text{ diagonal.}
\]

**Important**
If \( A = PDP^{-1} \) for \( D = \begin{pmatrix}
  d_{11} & 0 & \cdots & 0 \\
  0 & d_{22} & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & d_{nn}
\end{pmatrix} \), then

\[
A^k = PD^kP^{-1} = P \begin{pmatrix}
  d_{11}^k & 0 & \cdots & 0 \\
  0 & d_{22}^k & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & d_{nn}^k
\end{pmatrix} P^{-1}.
\]

So diagonalizable matrices are *easy to raise to any power*. 
The Diagonalization Theorem

An \( n \times n \) matrix \( A \) is diagonalizable if and only if \( A \) has \( n \) linearly independent eigenvectors.

In this case, \( A = PDP^{-1} \) for

\[
P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}
\]

\[
D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}
\]

where \( v_1, v_2, \ldots, v_n \) are linearly independent eigenvectors, and \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the corresponding eigenvalues (in the same order).

Important

- If \( A \) has \( n \) distinct eigenvalues then \( A \) is diagonalizable.

- If \( A \) is diagonalizable matrix it need not have \( n \) distinct eigenvalues though.
Problem: Diagonalize \( A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \).
Diagonalization
Example 2

Problem: Diagonalize \( A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \).
Diagonalization
Example 2, continued

Now let's compute the 2-eigenspace:

\[
(A - 2I)x = 0 \iff \begin{bmatrix}
2 - 3 & 0 \\
2 - 3 & 0 \\
1 & -1 -1
\end{bmatrix} x = 0
\]

\[
\text{rref} \begin{bmatrix}
1 & 0 \\
-3 & 1 \\
0 & 0 & 0
\end{bmatrix} x = 0
\]

The parametric form is \(x = 3z, y = 2z\), so an eigenvector with eigenvalue 2 is \(v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}\).

Note that \(v_1, v_2\) form a basis for the 1-eigenspace, and \(v_3\) has a distinct eigenvalue. Thus, the eigenvectors \(v_1, v_2, v_3\) are linearly independent and the Diagonalization Theorem says

\[
A = PDP^{-1}
\]

In this case: there are 3 linearly independent eigenvectors and only 2 distinct eigenvalues.
Diagonalization
Procedure

How to **diagonalize a matrix** $A$:

1. **Find the eigenvalues** of $A$ using the characteristic polynomial.
2. **Compute a basis** $B_\lambda$ for each $\lambda$-eigenspace of $A$.
3. If there are fewer than $n$ total vectors in the union of all of the eigenspace bases $B_\lambda$, then the matrix is not diagonalizable.
4. **Otherwise**, the $n$ vectors $v_1, v_2, \ldots, v_n$ in your eigenspace bases are linearly independent, and $A = PDP^{-1}$ for

   $$P = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

   and

   $$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

   where $\lambda_i$ is the eigenvalue for $v_i$. 
Problem: Show that \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) is not diagonalizable.

Conclusion:

- All eigenvectors of \( A \) are multiples of \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).
- So \( A \) has only one linearly independent eigenvector.
- If \( A \) was diagonalizable, there would be \textit{two linearly independent eigenvectors}!
Poll

Which of the following matrices are diagonalizable, and why?

A. \[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\]

B. \[
\begin{pmatrix}
1 & 2 \\
0 & 2
\end{pmatrix}
\]

C. \[
\begin{pmatrix}
2 & 1 \\
0 & 2
\end{pmatrix}
\]

D. \[
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]

Matrix D is already diagonal!

Matrix B is diagonalizable because it has two distinct eigenvalues.

Matrices A and C are not diagonalizable: All eigenvectors are multiples of \[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\].
Definition
Let $\lambda$ be an eigenvalue of a square matrix $A$. The geometric multiplicity of $\lambda$ is the dimension of the $\lambda$-eigenspace.

Theorem
Let $\lambda$ be an eigenvalue of a square matrix $A$. Then
\[ 1 \leq (\text{the geometric multiplicity of } \lambda) \leq (\text{the algebraic multiplicity of } \lambda). \]

- **Note:** If $\lambda$ is an eigenvalue, then the $\lambda$-eigenspace has dimension at least 1.
- **...but it might be smaller than what the characteristic polynomial suggests.** The intuition/visualisation is beyond the scope of this course.
Non-Distinct Eigenvalues

(Good) examples

From *previous exercises* we know:

**Example**

The matrix

\[
A = \begin{pmatrix}
4 & -3 & 0 \\
2 & -1 & 0 \\
1 & -1 & 1
\end{pmatrix}
\]

has characteristic polynomial

\[
f(\lambda) = -(\lambda - 1)^2(\lambda - 2).
\]

The matrix

\[
B = \begin{pmatrix}
1 & 2 \\
-1 & 4
\end{pmatrix}
\]

has characteristic polynomial

\[
f(\lambda) = (1 - \lambda)(4 - \lambda) + 2 = (\lambda - 2)(\lambda - 3).
\]

<table>
<thead>
<tr>
<th>Matrix $A$</th>
<th>Geom. M.</th>
<th>Alg. M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 1$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$\lambda = 2$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Matrix $B$</th>
<th>Geom. M.</th>
<th>Alg. M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\lambda = 3$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Thus, *both matrices are diagonalizable*. 
Non-Distinct Eigenvalues

(Bad) example

Example

The matrix \( A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) has characteristic polynomial \( f(\lambda) = (\lambda - 1)^2 \).

We showed before that the 1-eigenspace has dimension 1 and \( A \) was not diagonalizable.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>Geometric</th>
<th>Algebraic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda = 1 )</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

The Diagonalization Theorem (Alternate Form)

Let \( A \) be an \( n \times n \) matrix. The following are equivalent:

1. \( A \) is diagonalizable.
2. The sum of the geometric multiplicities of the eigenvalues of \( A \) equals \( n \).
3. The sum of all algebraic multiplicities is \( n \). And for each eigenvalue, the geometric and algebraic multiplicity are equal.
Applications to Difference Equations

Let \( D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \).

Start with a vector \( v_0 \), and let \( v_1 = Dv_0 \), \( v_2 = Dv_1 \), \ldots, \( v_n = D^n v_0 \).

Question: What happens to the \( v_i \)'s for different starting vectors \( v_0 \)?

- the \( x \)-coordinate equals the initial coordinate,
- the \( y \)-coordinate gets halved every time.
\[ D \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b/2 \end{pmatrix} \]

So all vectors get “collapsed into the x-axis”, which is the 1-eigenspace.
Applications to Difference Equations

More complicated example

Let \( A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix} \).

\( \text{Start with a vector } \mathbf{v}_0, \text{ and let } \mathbf{v}_1 = A\mathbf{v}_0, \mathbf{v}_2 = A\mathbf{v}_1, \ldots, \mathbf{v}_n = A^n\mathbf{v}_0. \)

Question: What happens to the \( \mathbf{v}_i \)'s for different starting vectors \( \mathbf{v}_0 \)?

\textbf{Matrix Powers:} This is a diagonalization question. \textbf{Bottom line: } A = PDP^{-1} \text{ for }

\[ P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}. \]

Hence \( \mathbf{v}_n = PD^nP^{-1}\mathbf{v}_0. \)
$A^n = PD^nP^{-1}$ acts on the usual coordinates of $v_0$ in the same way that $D^n$ acts on the $B$-coordinates, where $B = \{w_1, w_2\}$.

So all vectors get "collapsed into the 1-eigenspace".
Why is the Diagonalization Theorem true?

Suppose $A = PDP^{-1}$, where $D$ is diagonal with diagonal entries $\lambda_1, \lambda_2, ..., \lambda_n$. Let $v_1, v_2, ..., v_n$ be the columns of $P$. They are linearly independent because $P$ is invertible. So $Pe_i = v_i$, hence $P^{-1}v_i = e_i$.

$Av_i = PDP^{-1}v_i = PDe_i = P(\lambda_i e_i) = \lambda_i Pe_i = \lambda_i v_i$.

Hence $v_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$. So the columns of $P$ form $n$ linearly independent eigenvectors of $A$, and the diagonal entries of $D$ are the eigenvalues.

$A$ has $n$ linearly independent eigenvectors implies $A$ is diagonalizable: Suppose $A$ has $n$ linearly independent eigenvectors $v_1, v_2, ..., v_n$, with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. Let $P$ be the invertible matrix with columns $v_1, v_2, ..., v_n$. Let $D = P^{-1}AP$.

$De_i = P^{-1}AP e_i = P^{-1}Av_i = P^{-1}(\lambda_i v_i) = \lambda_i P^{-1}v_i = \lambda_i e_i$.

Hence $D$ is diagonal, with diagonal entries $\lambda_1, \lambda_2, ..., \lambda_n$. Solving $D = P^{-1}AP$ for $A$ gives $A = PDP^{-1}$. 

\[ A = PDP^{-1} \]