Homework 2

1. Suppose $C$ is the Cantor set. Show that

$$\{x + y, x \in C, y \in C\} = [0, 2].$$

2. Prove that if $X$ is a totally bounded metric space, then $X$ is separable, i.e. $X$ has a countable dense subset.

3. Let $A$ be the set of all sequences $(x_1, x_2, \ldots)$ that are eventually 0. What is the closure of $A$ in $\mathbb{R}^N$ in the product topology? In the box topology?

4. Show that no two of the space $(0, 1)$, $[0, 1]$ and $(0, 1]$ are homeomorphic. (Hint: What happens if you remove a point from each of these spaces?)

5. Show that a finite union of compact subspaces of $X$ is compact.

6. Suppose that $A$ is a closed subset of $X \times Y$, where $Y$ is compact, and $\pi_1 : X \times Y \to X$ is the projection onto the first component. Show that $\pi_1(A)$ is closed.

7. Show that the metric space $(X, d)$ is complete if and only if for every nested sequence $A_1 \supset A_2 \supset \ldots$ of non-empty closed sets of $X$ such that the diameter of $A_n \to 0$, the intersection of the sets $A_n$ is non-empty.