1. Let $\mathcal{H}$ be the following subset of $\ell^2$:

$$\mathcal{H} = \{(x_1, x_2, \ldots) \text{ such that } |x_n| \leq 1/2^n\}.$$ 

a) Show that $\mathcal{H}$ is convex.

b) Show that $\mathcal{H}$ does not have any internal point. (Hint: Suppose $x \in \mathcal{H}$, and $a = (1, 1/2, 1/3, \ldots)$, when $x + ta \in \mathcal{H}$?)

2. Suppose $V$ is a normed space and $f : V \to \mathbb{R}$ is a linear functional.

a) Show that $f$ is continuous if and only if $\ker(f)$ is closed.

b) Suppose $f$ is continuous. Let $H = f^{-1}(1)$. Show that 

$$d(0, H) = 1/||f||.$$ 

3. Let $C[0, 1]$ be the set of continuous functions on $[0, 1]$ with norm $||f|| = \max_{x \in [0,1]} |f(x)|$. Define

$$T(f) = af(0) + bf(1),$$

$$S(f) = \int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt.$$ 

Show that both $T, S$ are continuous linear functionals, and find their norms.

4. Suppose $A$ is a convex subset of $\mathbb{R}^n$ such that $\text{Aff}(A) = \mathbb{R}^n$. Show that $\text{int}(A) = \text{int}(\overline{A})$ and $\overline{A} = \text{int}(\overline{A})$. Give examples to show that these might not be true when $A$ is not convex.

5. Suppose $A, B \subset V$ are subsets of a linear spaces. Show that $A, B$ can be separated by a hyperplane if and only if $A - B$ and $\{0\}$ can be separated by a hyperplane.

6. For $v = (x, y) \in \mathbb{R}^2$ define $p(v) = (\sqrt{x} + \sqrt{y})^2$. Show that $p$ is not a norm for $\mathbb{R}^2$. (Hint: Is the unit ball convex?)