THE COLORED JONES POLYNOMIAL AND THE AJ CONJECTURE

par

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Résumé. — We present the basics of the colored Jones polynomial and discuss the AJ conjecture which relates the Jones polynomial and the A-polynomial of a knot.

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1. Introduction

In this note we give a survey of the theory of the colored Jones polynomials and the AJ conjecture which relates the Jones polynomial and the A-polynomial.

The Jones polynomial was discovered in 1984. It came as a shocking surprise in low-dimensional topology and has since stimulated many new developments. The Jones polynomial also opened new connections between knot theory and many other branches of mathematics and theoretical physics, such as Lie theory, number theory, and statistical physics. New algebraic structures are constructed in the study of the Jones polynomials. Soon after the discovery of the Jones polynomial, many generalizations, known as quantum invariants of knots and 3-manifolds, were discovered. In particular, for every simple Lie algebra $\mathfrak{g}$ and every finite-dimensional irreducible $\mathfrak{g}$-module, the theory assign to every knot in the 3-space an invariant, which is a Laurent polynomial in the quantum parameter. The colored Jones polynomial, which is an invariant of knots colored by integers, is among these generalizations ; it is the invariant corresponding to the Lie algebra $sl_2(\mathbb{C})$ and its finite-dimensional irreducible modules.

The Jones polynomial of a knot and its generalizations are defined through a diagram of the knot, an object essentially 2-dimensional. It is hard to understand the Jones polynomial in terms of classical invariants like the fundamental group, which is intrinsic 3-dimensional. The best known relation between

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the colored Jones polynomial and the fundamental group is the Melvin-Morton conjecture (now a theorem, see Subsection 4.6), which relates the colored Jones polynomial to the Alexander polynomial. The famous volume conjecture would connect the colored Jones polynomial to the hyperbolic structure of the knot complement. The Alexander polynomial is an *abelian* invariant of the knot complement, since it can be defined using abelian representations of the knot group. A finer invariant, the two variable A-polynomial, is defined using non-abelian representations of the knot group and its peripheral system. The A-polynomial has been important in geometric topology. The AJ conjecture would relate the colored Jones polynomial to the A-polynomial.

The goal of this note to give a friendly introduction to the colored Jones polynomial, to explain the AJ conjecture, and to sketch a proof of the AJ conjecture for a class of knots which includes infinitely many two-bridge knots and all pretzel knots $(-2, 3, 6n \pm 1)$.

In Section 2 we define the Jones polynomial through the Kauffman bracket and give a proof (due to Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links. In Section 3 we give an overview of quantum link invariants coming from quantum groups associated to simple Lie algebras. Section 4 is devoted to properties of the colored Jones polynomial, the Melvin-Morton conjecture, and the growth of the colored Jones polynomial. In Section 5 we show that for every knot, the color Jones function satisfies a recurrence relation, and we define the recurrence polynomial. In Section 6 we explain the Kauffman bracket skein module and its relation to character varieties. Section 7 is devoted to the AJ conjecture.

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2. The Jones polynomial

In this section we give the definition of the Jones polynomial via the Kauffman bracket, establish its basic properties, and sketch a proof (due to Kauffman, Murasugi, and Thistlethwaite) of the Tait conjecture on the crossing number of alternating links. The estimate of the degree bounds found in the proof of the Tait conjecture will be used in later sections. All the results in this section are now classic, and can be found for examples in textbooks [**Li, Oh**].

2.1. Knots and links in $\mathbb{R}^3 \subset S^3$. — Fix the standard 3-dimensional space $\mathbb{R}^3$. *An oriented link L* is a compact 1-dimensional oriented smooth submanifold of $\mathbb{R}^3 \subset S^3$. A link of 1 component is called a *knot.* By convention, the empty set is also considered a link.

A *framed oriented link L* is a link equipped with a smooth normal vector field V, which is a function $V : L \to \mathbb{R}^3$, such that $V(x)$ is not in the tangent space $T_x L$ for every $x \in L$.

Two (framed) oriented links are *equivalent* if one can be smoothly deformed into another in the class of (framed) oriented links.

A (framed) oriented link is *ordered* if there is an order on the set of its components.

Usually we don't distinguish between a link and its equivalence class. Un-oriented links, un-oriented framed links and their equivalence classes are defined similarly.

A link invariant is a map

$$I : \{\text{equivalence classes of links}\} \to S,$$

where S is a set.

Example 2.1. — For unoriented unframed links, the link group $\pi_1(L) := \pi_1(\mathbb{R}^3 \setminus L)$ is a link invariant.

2.2. Link diagram, blackboard framing. — One often studies an (oriented or unoriented) link L by studying one of its diagrams on $\mathbb{R}^2$, which is a projection D of L onto $\mathbb{R}^2$ (in general position), together with the "over/under" information at each crossing point. An (oriented) link diagram D of a link L determines the equivalence class of the (oriented) link L. Link diagrams are considered up to isotopy of the plane $\mathbb{R}^2$.

A link diagram comes with the *blackboard framing*, in which the framing vectors are in the plane $\mathbb{R}^2$. We say that a link diagram D is a *blackboard diagram* of a framed link L if the framed link determined by D together with its blackboard framing is equivalent to L.

It is known that two unoriented link diagrams define the same equivalence class of unoriented unframed links if and only if they are related by a sequence of Reidemeister moves RI, RII, and RIII (and isotopies of the plane). The Reidemeister moves are listed in Figure 1 and 2. For framed unoriented link diagrams one replaces RI by RI_f. For oriented links one allows all possible orientations of the strands in the figures. For details, see e.g [**BZ, Oh**].

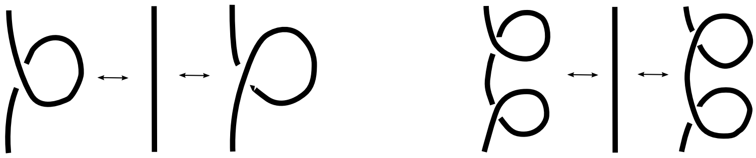


FIGURE 1. Reidemeister move RI on the left and RI_f on the right.

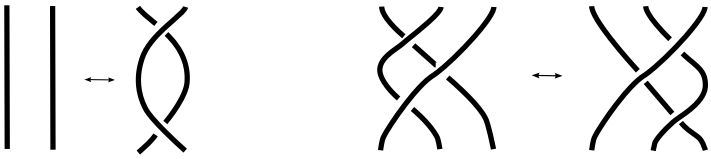


FIGURE 2. Reidemeister move RII on the left and RIII on the right.

Thus, the map associating an unoriented unframed link diagram to its link class descends to an isomorphism of sets

$$\{\text{link diagrams}\}/(\text{RI,RII,RIII}) \xrightarrow{\cong} \{\text{equiv. classes of links}\}.$$

If I is an invariant of unoriented link diagrams which is invariant under Reidemeister moves, then I descends to an invariant of unoriented unframed links.

The mirror image of a (framed, oriented) link L, denoted by $L!$, is the image of L under a reflection in a plane in $\mathbb{R}^3$. It is easy to see that the equivalence class of $L!$ depends only on the equivalence class of the original link L. If L has a (blackboard) framing D, then $L!$ has as a link diagram the *mirror image* of D, which is the result of switching all the crossings of D from over to under and vice versa.

2.3. Sign of a crossing, linking number, writhe. — Up to isotopies of the plane $\mathbb{R}^2$ there are two types of crossings of oriented link diagrams, see Figure 3. The crossing on the left is called a positive

FIGURE 3. A positive crossing and a negative crossing

crossing, while the one on the right is called a negative crossing.

For a 2-component oriented link diagram $D = D_1 \cup D_2$, define

$$\mathrm{lk}(D) = \frac{1}{2}\sum_x \varepsilon(x),$$

where the sum is over all the crossings between D_1 and D_2, and $\varepsilon(x)$ is the sign of x.

Exercise 2.2. — (a) Show that $\mathrm{lk}(D)$ does not change under oriented Reidemeister moves and hence defines an invariant of 2-component oriented links, known as the *linking number*.

(b) Suppose $L = L_1 \cup L_2$ be a 2-component oriented link. Define the Gauss map

$$\gamma : L_1 \times L_2 \to S^2 = \{z \in \mathbb{R}^3 \mid ||z|| = 1\}, \quad \gamma(x, y) = \frac{x - y}{||x - y||}.$$

Show that up to sign, $\mathrm{lk}(L_1, L_2)$ is equal to the degree of γ.

Let K be a framed unoriented knot in $\mathbb{R}^2$. Using the framing, one can push K off itself to get a *parallel* of K, which is well-defined up to isotopy in $\mathbb{R}^3 \setminus K$. An orientation of K induced an orientation of its parallel K', and the linking number $\mathrm{lk}(K, K')$ can be defined. It is easy to see that $\mathrm{lk}(K, K')$ does not depend on the choice of the orientation of K.

Exercise 2.3. — Suppose K is an unframed unoriented knot. Let $\mathrm{fr}(K)$ be the set of all framings of K. Show that the map $\mathrm{fr}(K) \to \mathbb{Z}$ given by $K \to \mathrm{lk}(K, K')$, where K' is a parallel of K determined by the framing, is a bijection.

As a result, we can, and often, use integers to denote framings of a knot.

Suppose D is the blackboard diagram of a framed oriented link. Define the *writhe* of L by

$$w(L) := \sum_{x \in C(D)} \varepsilon(x),$$

where $C(D)$ is the set of crossings of D.

Exercise 2.4. — (a) Show that $w(L)$ is an invariant of framed oriented links.

(b) Show that $w(L)$, when L is a framed knot, is the integer-valued framing of L, and does not depend on the orientation of the knot.

2.4. Alexander polynomial : homological definition. — Suppose L is an m-component oriented link, and $X = S^3 \setminus L$. A small loop encircling the j-th component is called a meridian of the component, which is defined up to isotopy in the link complement. We choose the orientation of the meridians so that the linking number of the j-th component and its meridian is $+1$.

From the Alexander duality, $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^n$, with generators being the meridians of the links. The map $H_1(X, \mathbb{Z}) \to \mathbb{Z} = \langle t \rangle$, mapping each meridian to t, gives rise to a surjective map $f : \pi_1(X) \to \mathbb{Z}$. The corresponding covering $\tilde{X} \to X$ has $\mathbb{Z}$ as the group of deck transformations. As a result, $H_1(\tilde{X}, \mathbb{Q})$ is a $\mathbb{Q}[\mathbb{Z}] \equiv \mathbb{Q}[t^{\pm 1}]$-module. Note that $\mathbb{Q}[t^{\pm 1}]$ is a principal ideal domain (PID), and $H_1(\tilde{X}, \mathbb{Q})$ is finitely generated over $\mathbb{Q}[t^{\pm 1}]$ (prove this !). According to the theory of finitely generate modules over a PID, we have

$$H_1(\tilde{X}, \mathbb{Q}) \cong \bigoplus_{j=1}^{k} \mathbb{Q}[t^{\pm 1}]/(f_j),$$

where $f_j \in \mathbb{Q}[t^{\pm 1}]$, $f_j | f_{j+1}$. Some of the f_j might be 0. The Alexander polynomial $\Delta_L(t) \in \mathbb{Q}[t^{\pm 1}]$ of L is defined to be $\prod_{j=1}^{k} f_k$.

The Alexander polynomial is defined up to a unit in $\mathbb{Q}[t^{\pm 1}]$. One can choose the unit normalization such that $\Delta_L(t) \in \mathbb{Z}[t^{\pm 1}]$.

If L is a knot, one can choose a unit normalization of Δ such that

$$\Delta_L(t^{-1}) = \Delta_L(t)$$

and $\Delta_L(1) = 1$, see Subsection 3.2. In particular, for any knot, $\Delta_L(t) \neq 0$. With this normalization, the Alexander polynomial of a knot is unique.

Exercise 2.5. — Let H be the Hopf link, see Figure 6. Show that $S^3 \setminus H$ is homotopic to the 2-dimensional torus $\mathbb{T}^2$. Show that $\Delta_H(t) = 1 - t$. On the other hand if L is the trivial two component link, then $S^3 \setminus L$ is homotopic to a bouquet of 2 circles. It follows that $\Delta_L(t) = 0$.

One can calculate the Alexander polynomial of an oriented link, beginning with a presentation of its fundamental group as follows. Suppose

$$\pi_1 = \langle a_1, \dots, a_{k+1} \mid r_1, \dots, r_k \rangle$$

is a Wirtinger presentation (see e.g. [**BZ**]) of the link group $\pi_1 = \pi_1(S^3 \setminus L)$. Then all a_j are conjugate to each other, and there is a group homomorphism

$$\mathrm{ab} : \pi_1 \to \mathbb{Z} = \langle t \rangle$$

given by $\mathrm{ab}(a_j) = t$ for $j = 1, \dots, k+1$. Let Y be the 2-dimensional CW-complex associated with the above mentioned group presentation, i.e. Y has one zero-cell, $k+1$ one-cells $a_1, \dots, a_{k+1}$, and k two-cells $b_1, \dots b_k$, such that the boundary of b_j, considered as an element of the free group generated by $\{a_1, \dots, a_{k+1}\}$, is equal to r_j. It is known that $X = S^3 \setminus L$ is homotopic to Y. Let $\frac{\partial r_j}{\partial r_j}$ be the Fox derivative. We consider $\frac{\partial r_i}{\partial r_j}$ as an element of $\mathbb{Z}[\pi_1]$. Let A be the $k \times k$ matrix $\left(\frac{\partial r_j}{\partial r_j}\right)_{i,j=1}^{k}$. By calculating the homology of the covering of Y associated to the map $\mathrm{ab} : \pi_1 \to \mathbb{Z}$, one can prove that $\Delta_K(t) = \det(\mathrm{ab}(A))$.

2.5. Kauffman bracket. — One of the best ways to define the Jones polynomial is to use the Kauffman bracket, introduced by [**Kau1**].

There is a unique function

$$\{\text{unoriented link diagrams}\} \to \mathbb{Z}[t^{\pm 1}], \quad D \to \langle D \rangle$$

satisfying

$$\langle D \rangle = t \langle D_+ \rangle + t^{-1} \langle D_- \rangle \tag{1}$$

$$\langle D \sqcup U \rangle = -(t^2 + t^{-2}) \langle D \rangle, \tag{2}$$

where in the first identity, D, D_+, D_- are identical except in a disk in which they look like in Figure 4, and in the second identity, the left hand side stands for the union of a diagram D and the trivial diagram

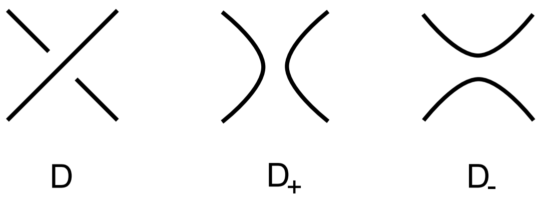
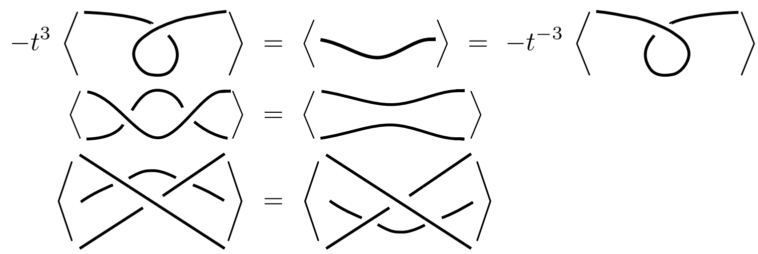


Figure 4. The diagrams D, D_+, and D_-

U disjoint from D. Here D might be the empty link diagram. In particular, if U is the unknot diagram, then

$$\langle U \rangle = -(t^2 + t^{-2}).$$

Lemma 2.6. — *One has*



Exercise 2.7. — Prove the lemma.

The lemma tells us that the Kauffman braket is invariant under the framed Reidemeister moves and hence defines an invariant of framed unoriented link.

Corollary 2.8. — *There exists a unique invariant*

$$\{\textit{oriented framed links}\} \to \mathbb{Z}[q^{\pm 1/4}], \quad L \to V_L \in \mathbb{Z}[q^{\pm 1/4}]$$

such that

$$q^{1/4} V_{L_+} - q^{-1/4} V_{L_-} = (q^{1/2} - q^{-1/2}) V_{L_0} \tag{3}$$

$$V_{L \sqcup U} = [2] V_L \tag{4}$$

$$V_{L^{+1}} = q^{3/4} V_L \tag{5}$$

Here, in (3)*, the links L_+, L_-, L_0 are identical everywhere except for a small ball in which they look like in Figure 5. In* (4)*, $L \sqcup U$ is the union of L and a trivial 0-framed knot U which is far away from L. In* (5)*, L^{+1} is the same as L, with the framing of one of the components increased by $+1$.*

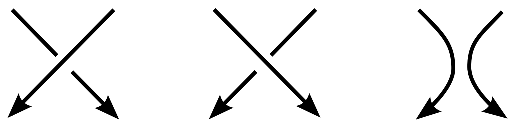


FIGURE 5. From left to right : the links L_+, L_- and L_0 in Equation (3)

Here we used the notation $[n]$ for the quantum integer (with $t = -q^{1/4}$)

$$[n] := \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}}.$$

Sketch of Proof. — By induction on the number of crossings one can show the uniqueness of V_L, using the relations (3), (4), and (5).

To show the existence, we will define V_L as follows. Suppose L is an oriented framed link with blackboard diagram D. Then

$$V_L(q) := (-1)^{\# L} \langle D \rangle \Big|_{t=-q^{1/4}}$$

is an invariant of link diagrams which is invariant under the framed Reidemeister moves and defines an invariant of L. It is not difficult to verify that V_L satisfying the requirements of the corollary. □

A simple normalization will give an invariant of unframed, oriented links. Let $\overset{\circ}{V}_L := q^{-(3/4)w(L)} V_L$, then $\overset{\circ}{V}$ is an invariant of oriented unframed links satisfying

$$q \overset{\circ}{V}_{L_+} - q \overset{\circ}{V}_{L_-} = (q^{1/2} - q^{-1/2}) \overset{\circ}{V}_{L_0} \tag{6}$$

$$\overset{\circ}{V}_{L \sqcup U} = [2] \overset{\circ}{V}_L \tag{7}$$

Remark 2.9. — The invariant $\overset{\circ}{V}_L$ is a version of the Jones polynomial [**Jo**].

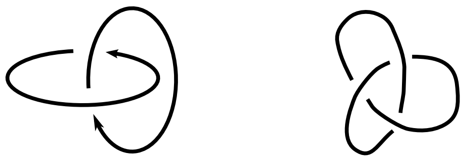


FIGURE 6. The Hopf link (on the left) and the right-handed trefoil

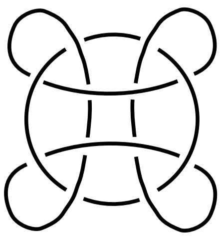


FIGURE 7. Milnor's link

2.6. Examples. — If L is the Hopf link of Figure 6, then

$$V_L = (t^4 + t^{-4})[2] = (q + q^{-1})[2].$$

If L is the right handed trefoil with framing 3 (see Figure 6), then

$$V_L = (-t^{-7} + t^{-3} + t^5)[2].$$

Exercise 2.10. — (Kauffman) The Milnor link is given in Figure 7. In his famous paper on Milnor's mu invariants, Milnor challenged us to find more invariants to distinguish links. He gave this example : at that time he did not know how to show that this link is not the trivial link. Calculate the Jones polynomial of the Milnor link and show that it is not the trivial link.

We see that the Jones polynomial captures very "fine" topology of knots and links which we don't fully understand yet.

2.7. Properties of the Jones polynomial. —

Proposition 2.11. — *(a) For every framed oriented link L one has*

$$V_L(q)\Big|_{q^{1/4}=1} = 2^{\#L}.$$

In particular, $V_L \neq 0$.
(b) Suppose $L!$ is the mirror image of L, then

$$V_{L!}(q) = V_L(q^{-1}).$$

(c) Suppose L is the connected sum of knots L_1 and L_2. Then

$$[2]\, V_L = V_{L_1} V_{L_2}.$$

(d) Suppose L' is a Conway mutation of L, then

$$V_L = V_{L'}.$$

(e) Suppose L has n components. Then $\overset{\circ}{V}_L(q) \in q^{n/2}\mathbb{Z}[q^{\pm 1}]$.

For the definition of the Conway mutation, see e.g. [**Li**].

Exercise 2.12. — Prove the proposition.

It follows that if L is a knot and $\mathring{V}_L(q) \neq \mathring{V}_L(q^{-1})$, then L is not amphichiral. For example, using the Jones polynomial, one can easily show that the trefoil is not amphichiral.

2.8. State sum of the Kauffman bracket. — Let D be a c-crossing link diagram. Denote by $C = C(D)$ the set of crossings.

At a crossing $x \in C$, the two strands of L divide a small neighborhood of x into four regions, two of them are marked $+$ and two are marked $-$ as in the middle part of Figure 8. The rule is : if one

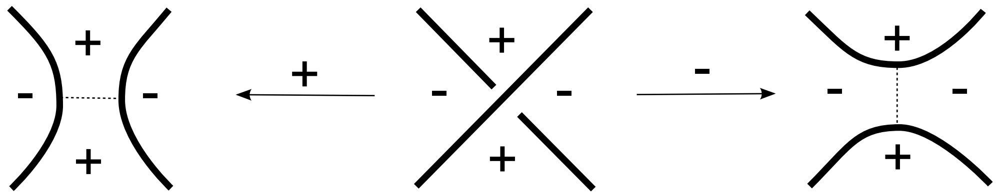


FIGURE 8. Positive resolution on the left and negative resolution on the right

rotates the over-crossing strand counterclockwise slightly, it will be in the two plus regions. There are two ways to resolve the singularity at x : the plus-resolution and the minus-resolution, see Figure 8. In the plus resolution, the two plus regions become connected (forget the dashed line). Similarly, in the minus resolution, the two minus regions become connected (forget the dashed line). In each resolution, we use a dashed line to connect the two resulting (solid) arcs.

A *state* for D is a function $s : C \to \{1, -1\}$. There are in total 2^c states. For a state s let sD be the diagram constructed from D by doing $s(x)$-resolution at every crossing x (without dashed lines). Then sD consists of disjoint simple closed curves on $\mathbb{R}^2$. Let $|sD|$ be the number of connected components of sD, and $\varsigma(s) = \sum_{c \in C} s(c)$.

Exercise 2.13. — Show that one always has $\varsigma(s) \equiv c := |C| \pmod 2$, for any state s.

Let G_s denote the graph whose vertices are connected components of sD and whose edges are the dashed arcs constructed above. Thus, G_s has $|sD|$ vertices and $c = |C|$ edges.

For a state s define

$$\langle s \rangle = t^{\varsigma(s)}(-t^2 - t^{-2})^{|sD|}. \tag{8}$$

From the definition of the Kauffman bracket, one has

$$\langle D \rangle = \sum_s \langle s \rangle. \tag{9}$$

2.9. Maximal degree and minimal degree. — For a non-zero polynomial $f \in \mathbb{Z}[t^{\pm 1}]$ let $\deg_+(f)$ and $\deg_-(f)$ be respectively the maximal degree and the minimal degree of non-zero monomials of f. The difference $\mathrm{br}(f) := \deg_+(f) - \deg_-(f)$ is called the *breadth* of the Laurent polynomial f.

For non-zero $f, g \in \mathbb{Z}[t^{\pm 1}]$, one has

$$\deg_+(fg) = \deg_+(f) + \deg_+(g), \ \deg_-(fg) = \deg_-(f) + \deg_-(g), \tag{10}$$

$$\mathrm{br}(fg) = \mathrm{br}(f) + \mathrm{br}(g) \tag{11}$$

$$\deg_+(f + g) \leq \max(\deg_+(f), \deg_+(g), \quad \text{if } f + g \neq 0. \tag{12}$$

We will try to find a state s_0 of D such that $\deg_+(\langle s_0 \rangle) > \deg_+(\langle s \rangle)$ for any state s other than s_0. Then Identity (9) shows that $\deg_+(\langle D \rangle) = \deg_+(\langle s_0 \rangle)$.

2.10. Partial order on states and monotonicity of $\deg_+, \deg_-$. — Fix an unoriented link diagram D with the set of crossing C. Recall that a state is a function $s : C \to \{-1, 1\}$. For two states s and s', we say $s \geq s'$ if $s(x) \geq s'(x)$ for every $x \in C$. This defines a partial order on the set of all states. The maximal state s_+ is the one which takes value 1 at every $x \in C$. Similarly, the minimal state s_- is the one which takes value -1 at every $x \in C$.

We say that s' is *one step below* s if $s' = s$ everywhere except for one crossing $x \in C$ where $s'(x) = -1$ and $s(x) = 1$.

Lemma 2.14. — *(a) Suppose $s \geq s'$. Then $\deg_+(s) \geq \deg_+(s')$.*

(b) Suppose s' is one step below s and $|sD| > |s'D|$. Then $\deg_+(s) > \deg_+(s')$.

Démonstration. — It is enough to consider the case when s' is one step below s. Then $\varsigma(s') = \varsigma(s) - 2$, and $|s'D|$ is either $|sD| - 1$ or $|sD| + 1$. Hence, from (8), one has $\deg_+(s) \geq \deg_+(s')$. Moreover, if, in addition, $|sD| > |s'D|$, then $\deg_+(s) > \deg_+(s')$. □

2.11. Adequate diagrams and breadth of Jones polynomial. — Since s_+ is the maximal state, from Lemma 2.14(a) we have

$$\deg_+(s_+) \geq \deg_+(s) \text{ for any state } s. \tag{13}$$

We want the strict inequality here. Lemma 2.14 shows that if $|s_+D| > |sD|$ for any state s which is one step below s_+, then $\deg_+(s_+) > \deg_+(s)$ for *any* state s other than s_+.

Definition 1. — *A link diagram D is plus-adequate if $|s_+D| > |sD|$ for any state s one step below s_+. A link diagram D is minus-adequate if $|s_-D| > |sD|$ for any state s one step above s_-. If both conditions hold, then D is called adequate.*

A link is plus-adequate (respectively minus-adequate, adequate) if it has a plus-adequate (respectively minus-adequate, adequate) diagram.

One can quickly recognize if a link diagram is adequate using part (c) or (d) of the following exercise.

Exercise 2.15. — Suppose D is a link diagram. Show that the following are equivalent.

(a) D is plus-adequate.

(b) The mirror image of D is minus adequate.

(c) At every crossing of D, the two arcs resulted in the positive resolution do not belong to the same connected component of s_+D.

(d) The graph G_{s_+} does not have any loop-edge.

We have seen that if D is plus-adequate, then $\deg_+(s_+)$ is strictly greater than $\deg_+(s)$ for *any* state s other than s_+. It follows from (9) that if D is plus-adequate, then

$$\deg_+(\langle D\rangle) = \deg_+(s_+D).$$

Similarly, if D is minus-adequate, then

$$\deg_-(\langle D\rangle) = \deg_+(s_-D).$$

The following is essentially due to Kauffman.

Theorem 2.16. — *Let D be a c-crossing link diagram. Then*

(a) $\deg_+(\langle D\rangle) \leq c + 2|s_+D|$, with equality if D is plus-adequate.

(b) $\deg_-(\langle D\rangle) \geq -c - 2|s_-D|$, with equality if D is minus-adequate.

Démonstration. — (a) Using (9) and (13), we have

$$\deg_+(\langle D\rangle) \leq \deg_+(\langle s_+\rangle) = c + |s_+D|,$$

where the last identity follows from Formula (8), with $\varsigma(s_+) = c$.

(b) The proof is similar. Alternatively, if one applies the result of (a) to the mirror image of D, then one gets (b). □

Suppose L is a plus-adequate framed unoriented link and D is a plus-adequate blackboard diagram of L. Then $\langle D\rangle$ is an invariant of L. It follows that $c(D) + 2|s_+D|$ is an invariant of plus-adequate framed unoriented links. This means, any two blackboard adequate link diagrams of L have the same $c + 2|s_+|$. To get c alone as an invariant of plus-adequate framed unoriented links, one can use parallels of links as follows.

For a link diagram D let D^n be the link diagram obtained from D by replacing each of its component by n of the component's parallels.

Lemma 2.17. — *Suppose D is a plus-adequate link diagram. Then D^n is also plus-adequate, with $c(D^n) = n^2c(D)$ and $|s_+(D^n)| = n|s_+(D)|$.*

The proof is easy and is left as an exercise.

Corollary 2.18. — *If L is a plus-adequate framed unoriented link and D is any adequate diagram of L. Then $c(D)$ and $|s_+(D)|$ are invariants of L.*

Démonstration. — From Theorem 2.16 and Lemma 2.17 we have that $c(D)n^2 + 2|s_+D|n$ is an invariant of L. Hence, $c(D)$ and $|s_+(D)|$ are invariants of L. □

Corollary 2.19. — *(a) Suppose L is an adequate framed unoriented link. Choose an adequate blackboard diagram D of L. Then $c(D)$, $|s_+D|$, and $|s_-D|$ are invariants of L.*

(b) Suppose L is an adequate unframed unoriented link. Choose an adequate blackboard diagram D of L. Then $c(D)$ is an invariant of L.

Démonstration. — (a) follows from Corollary 2.17.

(b) From Theorem 2.16, we have

$$\mathrm{br}(\langle D\rangle) = 2c(D) + 2(|s_+D| + 2|s_-D|).$$

Since D^n is an adequate link diagram (of L^n), we have

$$\mathrm{br}(\langle D^n\rangle) = 2c(D)n^2 + 2(|s_+D| + 2|s_-D|)n. \tag{14}$$

Suppose D' is another adequate diagram of L. Then $(D')^n$ is an adequate diagram of L. Let L_n be the framed link with blackboard diagram D^n and L'_n the framed link with blackboard diagram $(D')^n$. The left hand side of (14) is an invariant of L_n.

Then both L_n and L'_n have the same underlying unframed link L^n, which means they differ only by framings on components. It follows that $\mathrm{br}(V_{L_n}) = \mathrm{br}(V_{L'_n})$. Hence

$$2c(D)n^2 + 2(|s_+D| + 2|s_-D|)n = 2c(D')n^2 + 2(|s_+D'| + 2|s_-D'|)n$$

for every n. This implies $c(D) = c'(D)$ and $|s_+D| + |s_-D| = |s_+D'| + |s_-D'|$. □

Warning : There are knots which are both plus-adequate and minus-adequate, but not adequate. The reason is that the plus-adequacy and minus-adequacy might be realized by different diagrams.

2.12. Alternating links. — A link diagram is called *alternating* if along any component, the over/under nature of crossings is alternate. A link diagram D is *reduced* if it does not have a *removable crossing*, i.e. a crossing x for which there is an embedded disk in $\mathbb{R}^2$ whose boundary intersects D at exactly 2 points, both are near the x and belong to different strands, see Figure 9.

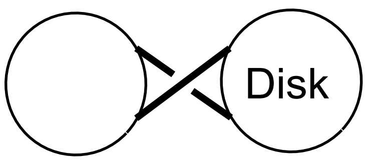


Figure 9. Removable crossing

Lemma 2.20. — *If an alternating link diagram D is reduced, then D is adequate.*

Démonstration. — Exercise. □

As a consequence, we have the following result.

Corollary 2.21. — *Suppose L is a an alternating unframed unoriented link. Then any two reduced diagrams of L have the same number of crossings.*

Exercise 2.22. — (a) Suppose D is a connected link diagram with c crossings. Then $|s_+D| + |s_-D| \leq c+2$, with equality if D is alternating.

(b) Suppose L is an alternating link and D, D' are diagrams of L, with D reduced alternating. Show that $c(D) \leq c(D')$.

Thus, if L is a link possessing an alternating diagram, then any two reduced alternating diagrams of L have the same number c of crossings, and this number c is minimum among all crossing numbers of diagram of L. With a little more effort one can also show that any non-alternating diagram of L has more than c crossings.

Exercise 2.23. — Suppose D is a connected non-trivial link diagram. Then the complement of D in S^2 consists of polygons. Each corner of every polygon is marked by $+$ or $-$, see Figure 8. Show that D is alternative if and only if the markings of all the corners of each region are the same.

3. Braid groups and link invariants

In this section we give an overview of quantum link invariants associated to a simple Lie algebra. We will define invariants of oriented links using the Markov theorem and Yang-Baxter operators. As examples we show how the Alexander polynomial and the Jones polynomial can be obtained through braids. We also survey main properties of quantum link invariants and discuss the case when the color of a knot is an infinite-dimensional module.

3.1. Braid groups and links. — *A braid in n-strands* is a compact 1-dimensional proper submanifold of $\mathbb{R}^2 \times 1$ consists of n strands such that its boundary is the set $\{1,2,\dots,n\} \times \{0\} \times \{0,1\}$ and such that no strand has critical points with respect to the vertical coordinate. Braids are considered up to isotopy of $\mathbb{R}^2 \times 1$ preserving the boundary and the vertical coordinate.

The set of all braids in n strands is denoted by $\mathfrak{B}_n$, which is a group where the product $\beta_1\beta_2$ of two braids is their concatenation, obtained by placing β_1 atop β_2. Let

$$\mathfrak{B} = \bigsqcup_{n=1}^{\infty} \mathfrak{B}_n,$$

be the set of all braids. One can define a tensor product in $\mathfrak{B}$ as follows. Suppose $\beta \in \mathfrak{B}_n$ and $\beta' \in \mathfrak{B}_m$. Let $\beta \otimes \beta'$ be the braid obtained by placing β' to the right of β.

The group $\mathfrak{B}_2$ is isomorphic to $\mathbb{Z}$ and generated by

(15) $\sigma =$  .

Let $\sigma_i \in \mathfrak{B}_n$ be defined by

$$\sigma_i = 1^{\otimes i-1} \otimes \sigma \otimes 1^{\otimes n-i-1}.$$

It is known that the group $\mathfrak{B}_n$ has a presentation

$$\mathfrak{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j|>1, \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \rangle.$$

The closure $\hat{\beta}$ of a braid β is the oriented link obtained from β by connecting upper and lower ends of β as in Figure 10, where the orientation is chosen so that on each strand of the braid it is pointing downward. Alexander shows that every oriented link is the closure of some braid. Markov shows that $\hat{\beta} = \hat{\beta}'$ if and only if β and β' are related by a finite number of the Markov moves (i)–(ii) :

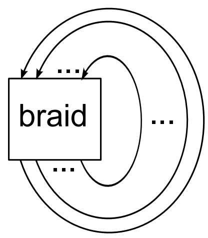


FIGURE 10. Oriented links as closures of braids

(i) $\beta\beta' \longleftrightarrow \beta'\beta$ for any $\beta, \beta' \in \mathfrak{B}_n$, and
(ii) $\beta \longleftrightarrow \beta\sigma_n^{\pm 1}$ for $\beta \in \mathfrak{B}_n$.
For details, see e.g. [**KT**].
In other words, the closing operator descends to a bijective map

$$\left(\bigsqcup_{n=1}^{\infty} \mathfrak{B}_n\right) / \{\text{Markov's moves}\} \xrightarrow{\cong} \{\text{oriented links}\}.$$

This means, if we can find a function $f : \mathfrak{B} = \bigsqcup_{n=1}^{\infty} \mathfrak{B}_n \to S$ which is invariant under the Markov moves, then f descends to an invariant of oriented links. Although Markov's theorem was known a long time ago (1938), the first non-trivial example of a link invariant constructed this way is the Jones polynomial (1984).

For a braid $\beta \in \mathfrak{B}_n$, the link group of $\hat{\beta}$ can be calculated as follows. By interpreting $\mathfrak{B}_n$ as the mapping class group of an n-punctured disk, one obtains an action of $\mathfrak{B}_n$ on the free group F_n on n generators $a_1, \dots, a_n$, see e.g. [**KT**]. The action is given by

$$\sigma_i(a_k) = \begin{cases} a_k a_{k+1} a_k^{-1} & \text{if } k = i, \\ a_{k-1} & \text{if } k = i+1, \\ a_k & \text{otherwise.} \end{cases}$$

The fundamental group of $\hat{\beta}$ has the presentation

$$\pi_1 = \langle a_1, \dots, a_n \mid a_i = \beta(a_i), i = 2, \dots, n\rangle. \tag{16}$$

Exercise 3.1. — Check that the group π_1 defined as above for braids is invariant under the Markov moves.

3.2. Burau representation, Alexander polynomial, and spanning trees. — A function on $\mathfrak{B}_n$ is invariant under the first Markov move if and only if it is a class function, i.e. a function constant on conjugacy classes. For example, if $\phi : \mathfrak{B}_n \to GL_k(\mathbb{C})$ is representation, then $f(\beta) = \det(\phi(\beta))$ is a class function.

The best known representation of the braid group is the Burau representation $\phi : \mathfrak{B}_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$ given by

$$\phi(\sigma_j) = I_{j-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-j-1},$$

where I_k is the $k \times k$ identity matrix.

Exercise 3.2. — (a) Check that the above formula gives a well-defined representation of the braid group.

(b) For $\beta \in \mathfrak{B}_n$ let $\Phi(\beta)$ be the $n \times n$ matrix with entries in $\mathbb{Z}[\pi_1(\hat{\beta})]$ defined by

$$\Phi(\beta) = \left(p\left(\frac{\partial \beta(a_i)}{\partial a_j}\right)\right)_{i,j=1}^{n}, \tag{17}$$

where $\mathfrak{B}_n$ acts on the free group $F_n = \langle a_1, \dots, a_n\rangle$ as decribed in the previous section, $\frac{\partial \beta(a_i)}{\partial a_j}$ is the Fox derivative, and $p : F_n \to \pi(\hat{\beta})$ is the natural projection map. Show that

$$\phi(\beta) = \mathrm{ab}(\Phi(\beta)),$$

where $\mathrm{ab} : \pi_1(\hat{\beta})) \to \mathbb{Z} = \langle t\rangle$ is the homomorphism defined by $\mathrm{ab}(a_i) = t$ for $i = 1, \dots, n$.

Let $w : \mathfrak{B}_n \to \mathbb{Z}$, known as the *writhe*, be the group homomorphism given by $w(\sigma_i) = 1$ for every $i = 1, 2, \dots, n-1$.

Theorem 3.3. — *The function $\nabla : \mathfrak{B} \to \mathbb{Z}[t^{\pm 1/2}]$ given by*

$$\nabla(\beta) = t^{\frac{1}{2}(n-w(\beta)-1)} \det(I_n - \phi(\beta))' \quad \textit{for} \ \ \beta \in \mathfrak{B}_n,$$

where $(I_n - \rho(\beta))'$ is the $(n-1)\times(n-1)$ matrix obtained by removing the first row and the first column from $I_n - \rho(\beta)$, is invariant under the Markov moves and hence defines an invariant $\nabla(\hat{\beta})$ of the oriented link $\hat{\beta}$.

The invariant $\nabla(L)$ satisfies the following skein relation

$$\nabla(U) = 1 \tag{18}$$

$$\nabla(L_+) - \nabla(L_-) = (t^{-1/2} - t^{1/2})\nabla(L_0), \tag{19}$$

where U is the trivial knots, and L_+, L_-, L_0 are any three links identical everywhere except in a ball where they look like in Figure 5.

The invariant $\nabla(L)$ is equal to the Alexander polynomial $\Delta(L)$, which is defined only up to $\pm t^m, m \in \mathbb{Z}$.

Sketch of Proof. — Let v_r be the $1\times n$ matrix, $(1, t, t^2, \dots, t^{n-1})$ and v_c be the $n\times 1$ matrix, $(1, 1, \dots, 1)^T$. We consider v_r as a row vector and v_c as a column vector. By checking with the generators σ_i one sees that v_r and v_c are respectively a left eigenvector and a right eigenvector of $\rho(\beta)$ for every $\beta \in \mathfrak{B}_n$, where the eigenvalue in both cases is 1. In other words,

$$v_r\,\rho(\beta) = v_r, \quad \rho(\beta)\,v_c = v_c.$$

As a consequence, letting $B = I_n - \phi(\beta)$, one has

(*) v_r is a left null-vector of B, i.e. $v_r\,B = 0$. Similarly, v_c is a right null-vector of B.

Let $C_{ij} = C_{ij}(B)$ be the (i,j)-cofactor of B, which is $(-1)^{i+j}\det(B_{(ij)})$, where $B_{(ij)}$ is the $(n-1)\times(n-1)$ submatrix of B obtained by removing the i-th row and the j-th column. Property (*) implies that all the cofactors are the same up to a power of t. More precisely,

$$C_{ij} = t^{n-j} C_{1,1}. \tag{20}$$

(Prove this !)

The "positive" characteristic polynomial of $B = B(\beta)$,

$$\det(\lambda + B) = \sum_{j=0}^{n} f_j \lambda^j,$$

is a class function. We have

$$f_1 = \sum_j C_{jj} = \left(\sum_j t^{n-j}\right) C_{1,1}.$$

It follows that $C_{1,1}$ is a class function. Hence $\nabla(\beta) = t^{(n-w(\beta)-1)/2} C_{1,1}$ is a class function.

By comparing $C_{nn}(\beta)$ and $C_{n+1,n+1}(\beta\sigma_n^{\pm 1})$ for $\beta \in B_n$ and using (20) it is easy to show that ∇ is also invariant under the second Markov moves and hence defines an oriented link invariant.

Suppose L_+, L_-, L_0 are as in the theorem. Then there is a braid $\beta \in \mathfrak{B}_n$ such that

$$L_+ = \widehat{\sigma_1\beta}, \quad L_- = \widehat{\sigma_1^{-1}\beta}, \quad L_0 = \widehat{\beta}.$$

A calculation of ∇ for the braids $\sigma_1\beta, \sigma_1^{-1}\beta, \beta$ will show that ∇ satisfy the skein relation (19). Identity (18) follows from an easy calculation with the trivial braid in $\mathfrak{B}_1$.

One can show that $\nabla(\hat{\beta})$ is equal to the Alexander polynomial along the following line. The fundamental group of $\pi_1(\hat{\beta})$ has presentation (16). The CW-complex $\mathcal{C}$ of the universal covering of the 2-dimensional CW-complex associated to the above presentation of π_1 has the form

$$\mathcal{C} \ = \ \left(0 \to (\mathbb{Z}[\pi_1])^{n-1} \xrightarrow{\partial_2} (\mathbb{Z}[\pi_1])^n \xrightarrow{\partial_1} \mathbb{Z}[\pi_1] \to 0\right). \tag{21}$$

where

$$\partial_1 = \begin{pmatrix} a_1 - 1 \\ \dots \\ a_n - 1 \end{pmatrix},$$

and ∂_2 is the $(n-1) \times n$ matrix obtained from $(I - \Phi(\beta))$ by removing the first row. Here $\Phi(\beta)$ is given by (17).

The homology H_1 of the abelian covering corresponding ab : $\pi_1 \to \mathbb{Z}$ is the $H_1(\mathcal{C}_{\text{ab}})$, where $\mathcal{C}_{\text{ab}} := \mathcal{C} \otimes_{\mathbb{Z}[\pi_1]} \mathbb{Z}[\mathbb{Z}]$. Because $\text{ab}(\Phi(\beta)) = \phi(\beta)$, one can show that the order of $H_1(\mathcal{C}_{\text{ab}})$, which is the Alexander polynomial of $\hat{\beta}$, is equal to $\nabla(\beta)$. This proves $\nabla_L = \Delta_L$. □

The polynomial $\nabla_L(t)$ is defined without ambiguity, unlike the original Alexander polynomial, which is defined up to a factor $\pm t^m, m \in \mathbb{Z}$.

Exercise 3.4. — Show that for every knot K one has $\nabla_K(1) = 1$. If L has at least two components, then $\nabla_L(1) = 0$.

Exercise 3.5. — Using the skein relation (19) show that for an oriented link L with m components,

$$\varphi(\nabla_L) = (-1)^{m-1} \nabla_L,$$

where φ is the algebra involution of $\mathbb{Z}[t^{\pm 1/2}]$ given by $\varphi(t^{1/2}) = t^{-1/2}$.

If one uses the homological definition of the Alexander polynomial, the above symmetry can be established using the duality of Reidemeister torsions.

It is amusing to compare the above definition of the Alexander polynomial with Kirchhoff's formula of the number of spanning trees of a graph. Suppose Γ is a connected graph, i.e. a connected 1-dimensional finite CW-complex. For simplicity assume that Γ is k-regular, i.e. the degree of every vertex of Γ is k. Let A be the adjacency matrix of Γ. The matrix $kI - A$, known as the Laplacian of the graph, has the property that the row vector $(1, 1, \dots, 1)$ and the column vector $(1, 1, \dots, 1)^T$ are respectively a left and a right null-vector. All the co-factors of $kI - A$ are same, and the common value of these co-factors, according to Kirchhoff's formula, is the number of spanning trees of Γ.

It would be interesting to find an interpretation of the Alexander polynomial in terms of spanning trees, with the Burau matrix being the "adjacency matrix". **Warning :** the Burau matrix is not symmetric in the usual sense, but it is "unitary", see [**KT**].

If we do not abelianize the matrix $\Phi(\beta)$, then we can get the volume of the link complement as follows. Remove any one row and one column of $\Phi(\beta)$ to get an $(n-1) \times (n-1)$ matrix $\Phi(\beta)'$ with entries in $\mathbb{Z}[\pi_1]$. As for any matrix with entries in $\mathbb{C}[\pi]$, there is defined the Fuglede-Kadison determinant $\det_{\pi_1}(I - \Phi(\beta)')$, see e.g. [**LS**]. From a result of Lück and Schick [**LS**] one can show that $\det_\pi(I - \Phi(\beta)') = \text{Vol}(\hat{\beta})/6\pi$ if $\hat{\beta}$ is a non-split link. Here $\text{Vol}(\hat{\beta})$ is the sum of the hyperbolic volume of the hyperbolic pieces of the JSJ decomposition of $S^3 \setminus \hat{\beta}$. Thus, one could probably also try to find an interpretation of the volume $\text{Vol}(\hat{\beta})$ in terms of number of spanning trees of graphs.

3.3. Yang-Baxter operator and link invariants. — The way Jones discovered his famous polynomial is, while studying subfactors of Von Neumann factors of type II_1, he found a representation of braid groups which can be used to define a link invariant via Markov's theorem. Let us now describe this approach, in a form which is slightly different from the original one of Jones'.

Suppose we have for each n a representation $\rho : \mathfrak{B}_n \to V^{\otimes n}$, where V is V is a vector space, say over $\mathbb{C}$. We say that ρ is *local* if $\rho(\beta \otimes \beta') = \rho(\beta) \otimes \rho(\beta')$. A local representation is thus totally determined by $R = \rho(\sigma)$, where $\sigma \in \mathfrak{B}_2$ is the braid in (15). From the defining relations of the braid group one sees that an invertible operator $R : V \otimes V \to V \otimes V$ defines a local representation if and only if

$$R_1 R_2 R_1 = R_2 R_1 R_2, \tag{22}$$

where R_1, R_2 are automorphisms of $V^{\otimes 3}$ defined by

$$R_1 = R \otimes \mathrm{id}, \quad R_2 = \mathrm{id} \otimes R.$$

An invertible $R : V \otimes V \to V \otimes V$ satisfying (22) is called a Yang-Baxter (YB) operator. We denote the local representation defined by a YB operator R by ρ_R.

Of course we always have $\mathrm{tr}(\rho(\beta\beta')) = \mathrm{tr}(\rho(\beta'\beta))$. That is, $\beta \to \mathrm{tr}(\rho(\beta))$ is invariant under the first Markov move. But how to deal with the second Markov move $\beta \sim \beta\sigma_n^{\pm}$, which relates an element in $\mathfrak{B}_n$ and an element in $\mathfrak{B}_{n+1}$? The locality nature of the representations allows us to reduce this problem to a local problem, and one solution is suggested by Turaev [**Tu1**] as follows.

An *enhanced YB operator* consists of a YB operator R and an invertible operator $\eta : V \to V$ such that $R(\eta \otimes \eta) = (\eta \otimes \eta)R$ and

$$R(\eta \otimes \eta) = (\eta \otimes \eta)R \tag{23}$$

$$\mathrm{tr}_2((\mathrm{id}_V \otimes \eta)R^{\pm 1}) = \mathrm{id}_V, \tag{24}$$

where $\mathrm{tr}_2(f)$, for and endomorphism $f : V^{\otimes 2} \to V^{\otimes 2}$, is the trace with respect to the second component, with the result being an endomorphism of V.

Fix an enhanced YB operator (R, η). For an endomorphism $f : V^{\otimes n} \to V^{\otimes n}$ define its quantum trace by

$$\mathrm{tr}_q(f) = \mathrm{tr}(\eta^{\otimes n} f).$$

The fact that $\eta \otimes \eta$ commutes with R ensures that $\beta \to \mathrm{tr}_q(\rho_R(\beta))$, for $\beta \in \mathfrak{B}_n$, is still a class function, i.e. invariant under the 1st Markov move. It is not difficult to show that (24) implies that $\mathrm{tr}_q(\rho_R(\beta))$ is invariant under the second Markov move. Thus, we the following.

***Theorem 3.6* (Turaev).** — *Suppose (R, η) is an enhanced Yang-Baxter operator. Then the map $\beta \to \mathrm{tr}_q(\rho_R(\beta))$, defined on $\mathfrak{B} = \sqcup_{n=1}^{\infty} \mathfrak{B}_n$, is invariant under the Markov moves and hence defines an invariant of oriented links.*

3.4. Jones polynomial from enhanced YB operator. — We give here the first non-trivial example of an enhanced YB operator, which gives the Jones polynomial. Let $V = \mathbb{C}^2$ with an ordered basis $\{e_0, e_1\}$. We use $e_0 \otimes e_0, e_0 \otimes e_1, e_1 \otimes e_1$ as an ordered basis of $V \otimes V$. Define $R : V \otimes V \to V \otimes V$ and $\eta : V \to V$, given in the mentioned bases by

$$R = \begin{pmatrix} q^{-1/2} & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & q^{-1} & q^{-1/2} - q^{-3/2} & 0 \\ 0 & 0 & 0 & q^{-1/2} \end{pmatrix}, \eta = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}.$$

Both R and η depend on a parameter $q^{1/2}$. We can consider $q^{1/2}$ as a non-zero complex number, or as a formal variable. The reader is invited to check that the pair (R, η) form an enhanced YB operator. Hence, if $\beta \in \mathfrak{B}_n$, then

$$\mathrm{tr}_q(\rho(\beta)) = \mathrm{tr}(\eta^{\otimes n} \rho(\beta)) \in \mathbb{Z}[q^{\pm 1/2}]$$

is an invariant of the oriented link $\hat{\beta}$, denoted by $W_{\hat{\beta}}$ for the moment.

Let us show that W_L is equal to the unframed version $\overset{o}{V}_L$ of the Jones polynomial. First we see that for the unknot U,

$$W_U = [2] = \overset{o}{V}_U.$$

The YB operator above, although a 4×4 matrix, has a quadratic minimal polynomial. More precisely,

$$qR - q^{-1}R^{-1} = (q^{1/2} - q^{-1/2})\,\mathrm{id}.$$

From here, we see that W_L satisfies the skein relation

$$qW_{L_+} - W_{L_-} = (q^{1/2} - q^{-1/2})W_{L_0}$$

which is the same relation (6) which $\overset{o}{V}_L$ satisfies. It follows that $W_L = \overset{o}{V}_L$.

In general, the invariant coming from an enhanced YB operator (R, η) will satisfy a k-term skein relation, where $k - 1$ is the degree of the minimal polynomial of R.

3.5. General quantum link invariant coming from semi-simple Lie algebras. — Works of Drinfeld, Jimbo, and others (see e.g. [**CP**], with our q equal to the q of [**CP**]), show that for every simple Lie algebra $\mathfrak{g}$, every irreducible finite-dimensional $\mathfrak{g}$-module V, there is an enhanced YB operator (R, η) acting on a certain vector space V_q over $\mathbb{Q}(q^{1/d})$, which is considered a q-deformation of V. Here d is twice the determinant of the Cartan matrix of $\mathfrak{g}$, and the dimension of V_q over $\mathbb{Q}(q^{1/d})$ is equal to the dimension of V over $\mathbb{C}$. Actually, there is a *quantized enveloping algebra* $U_q(\mathfrak{g})$, a Hopf algebra over $\mathbb{Q}(q^{1/d})$ which can be considered as a quantum deformation of $\mathfrak{g}$, such that V_q is an irreducible $U_q(\mathfrak{g})$-module. Direct sums of V_q's are called $U_q(\mathfrak{g})$-modules of type 1. Finite-dimensional irreducible $U_q(\mathfrak{g})$-modules of type 1 are parameterized by dominant weights of $\mathfrak{g}$.

The YB operator R commutes with the action of $U_q(\mathfrak{g})$, i.e. $R : V_q^{\otimes 2} \to V_q^{\otimes 2}$ is a $U_q(\mathfrak{g})$-morphism. It follows that $\rho(\beta)$ is a $U_q(\mathfrak{g})$-morphism for every $\beta \in \mathfrak{B}_n$.

As $U_q(\mathfrak{g})$ is a Hopf algebra, if V, V' are $U_q(\mathfrak{g})$-module, then there is a natural $U_q(\mathfrak{g})$-module structure on $V \otimes V'$, defined by the co-product of $U_q(\mathfrak{g})$.

The action of η on every $U_q(\mathfrak{g})$-module is given by an element in $U_q(\mathfrak{g})$, also denoted by η, and this element is group-like in the sense that its co-product is $\eta \otimes \eta$. It follows that the action of η on $\otimes_{i=1}^n(V_i)$ is given by $\eta^{\otimes n}$. The YB operator R also comes from a universal YB element, defined in a certain completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.

For any $\mathbb{Q}(q^{1/d})$-module homomorphism $f : V \to V$, define its *quantum trace* by

$$\mathrm{tr}_q(f) = \mathrm{tr}(f\eta).$$

When $f = \mathrm{id} : V \to V$, the quantum trace of f, $\mathrm{tr}_q(\mathrm{id})$, is called the quantum dimension of V.

As mentioned, for every irreducible finite-dimensional $\mathfrak{g}$-module V, there is an enhanced YB operator (R, η) acting on V_q. Hence we can define invariants of oriented links by

$$J_L(V) := \mathrm{tr}_q(\rho(\beta)) \in \mathbb{Q}(q^{1/d}),$$

where $L = \hat{\beta}$, and $\beta \in \mathfrak{B}_n$ is a braid.

When $\mathfrak{g} = sl_2$ and $V = \mathbb{C}^2$, the defining 2-dimensional representation of $sl_2(\mathbb{C})$, enhanced YB pair (R, η) are described in the previous section, and its link invariant if the Jones polynomial.

Suppose now $\mathfrak{g} = sl_n$ and $V = \mathbb{C}^n$, which is considered as an sl_n-module in the obvious way. The corresponding link invariant is denoted by $J_L^{sl_n}$. In this case the YB operator R, although a matrix of size $n^2 \times n^2$, satisfies a quadratic polynomial. In fact,

$$q^{n/2}R - q^{-n}R^{-1/2} = (q^{1/2} - q^{-1/2})\,\mathrm{id}\,.$$

It follows that the corresponding unframed oriented link invariant satisfies the skein relation

$$q^{n/2}J_{L_+}^{sl_n} - q^{-n/2}J_{L_-}^{sl_n} = (q^{1/2} - q^{-1/2})J_{L_0}^{sl_n},$$

where L_+, L_-, L_0 are identical everywhere except for a small balls in which they are as in Figure 5. By combining all the above invariants, setting q^n to be a new parameter, one gets the two variable HOMFLY-PT polynomial of links. The invariants associated to so_N and their fundamental representations can be used to define the Kauffman polynomial [**Kau2**]. For details, see [**Tu1**].

3.6. Different colors. — The above invariant is an invariant of pairs (L, V), where L is an unframed oriented links and V is an irreducible finite-dimensional $\mathfrak{g}$-module V. We call the pair (L, V) a *colored link*, with V the color of each component of L.

With a little more effort, the theory of ribbon category extends the above invariant to the case when each individual component of L has its own color, which is a finite-dimensional $\mathfrak{g}$-module. The theory also gives *operator invariants* of tangles. Orientation and framing are important here. Let us summarize the outcome of this theory, focusing only on string links. For details, see e.g. [**Tu4, Oh, CP**].

An n-component string link β is a compact 1-dimensional proper submanifold of $\mathbb{R}^2 \times 1$ consists of n strands such that the boundary of the i-th strand is the set $\{i\} \times \{0\} \times \{0,1\}$. Two string links are considered equivalent if one can be smoothly deformed into the other in the class of string links. The set of all n-component string links form a monoid, where the product TT' is obtained by placing T atop T'. For two string links, not necessarily of the same number of components, we also define the tensor product $T \otimes T$, which is obtained by placing T_2 to the right of T_1.

A framed string link is a string link equipped with a smooth normal vector field such that at every boundary point the normal vector is $(0,1,0)$. One defines equivalence classes and the monoids of framed string links similarly.

We will consider each string link as oriented, with the convention that the orientation of is chosen so that on each strand it is pointing downward near the boundary points. The closure $\hat{\beta}$ of a framed string link β is the framed oriented link obtained from β by connecting upper and lower ends of β as in the braid case, see Figure 10. Here the framing on the closing part is always standard, i. e. the normal vector of the framing here is always equal to $(0,1,0)$.

Suppose T is a framed string link with n components, and $V_1,\dots,V_n$ are finite-dimensional $U_q(\mathfrak{g})$-modules of type 1, not necessarily irreducible. According to the theory of ribbon category associated to the quantum group $U_q(\mathfrak{g})$, there is defined a $U_q(\mathfrak{g})$-morphism

$$J_T : \bigotimes_{j=1}^{n} V_j \to \bigotimes_{j=1}^{n} V_j,$$

such that

$$J_{TT'} = J_T J_{T'}, \quad J_{T\otimes T'} = J_T \otimes J_{T'}.$$

Actually, the machinery of quantum group defines J_T as an element of a certain completion $\widehat{U_q(\mathfrak{g})^{\otimes n}}$ of $U_q(\mathfrak{g})^{\otimes n}$. The completion $\widehat{U_q(\mathfrak{g})^{\otimes n}}$ acts on any finite-dimensional $U_q(\mathfrak{g})^{\otimes n}$-modules of type 1. This $J_T \in \widehat{U_q(\mathfrak{g})^{\otimes n}}$ is called the *universal invariant* of T, see e.g. [**Oh, Ha2**]. Besides, J_T commutes with the images of the co-product, which explains why the action of J_T on $\otimes_{j=1}^n V_j$ is a $U_q(\mathfrak{g})$-morphism. In particular, if T is a 1-component string link, then J_T is a central element in $\widehat{U_q(\mathfrak{g})}$.

If the framed oriented link L is the closure of T, then

$$J_L(V_1,\dots,V_n) = \mathrm{tr}_q^{V_1\otimes\cdots\otimes V_n}(J_T) := \mathrm{tr}(J_T\,\eta^{\otimes n}, V_1\otimes\cdots\otimes V_n)$$

is an invariant of the framed oriented link L whose j-th component is colored with V_j.

This invariant has the following properties

(0) Trivial color : Removing a link component colored by the trivial representation does not affect the value of the invariant.

(i) Integrality : For each framed oriented link L and colors $V_1,\dots,V_n$, there is $a \in \mathbb{Z}$ such that $J_L(V_1,\dots,V_m) \in q^{a/d}\mathbb{Z}[q^{\pm 1}]$, see [**Le1**].

(ii) Additivity : if $V = V' \oplus V''$ then $J_L(V,\dots) = J_L(V',\dots) + J_L(V'',\dots)$.

(iii) tensor product formula : If $V = V' \otimes V''$, then

$$J_L(V,\dots) = J_{L^{(2)}}(V',V'',\dots),$$

where $L^{(2)}$ is obtained from L by replacing the 1st component by two of its parallels (using the framing).

(iv) Orientation reversing formula : If L' is the same L with the orientation of the 1st component reversed, and V^* is the dual of V, then

$$J_{L'}(V,\dots) = J_L(V^*,\dots).$$

Thus, for Lie algebra for which $V \cong V^*$, for example, $\mathfrak{g} = sl_2(\mathbb{C})$ or $\mathfrak{g}$ is any simple Lie algebra of the B series, then the invariant is insensitive to the orientation of the components.

Suppose V is an irreducible $U_q(\mathfrak{g})$-module of type 1 and T is a 1-component string link. i.e. a long knot. By Schur's lemma, $J_T : V \to V$ is a scalar operator, i.e. $J_T = J'_T\,\mathrm{id}$, where $J'_T \in \mathbb{Q}(q^{1/d})$ is a scalar. It follows that

$$J_{\hat{T}}(V) = \mathrm{tr}_q^V(J_T) = J'_T \dim_q(V).$$

Suppose T_1, T_2 are 1-component string links, with $K_i = \hat{T}_i$ and $K = \widehat{T_1T_2}$. Then

$$J_K(V) = \mathrm{tr}_q^V(J_{T_1T_2}) = \mathrm{tr}_q^V(J_{T_1}J_{T_2}) = \mathrm{tr}_q^V(J'_{T_1}J'_{T_2}\,\mathrm{id}) = J'_{T_1}J'_{T_2}\dim_q(V).$$

Hence, we have the following.

Proposition 3.7. — *Suppose K_1, K_2 are framed oriented knots and K is the connected sum of K_1 and K_2. Assume V is an* irreducible *$\mathfrak{g}$-module, then*

$$J_K(V) = J_{K_1}(V)J_{K_2}(V)/\dim_q(V). \tag{25}$$

Suppose V is a *highest weight $U_q(\mathfrak{g})$-module* (see e.g. [**CP**]), and T is a 1-component string link. The universal invariant J_T is an element of certain extension $U_h(\mathfrak{g})$, which is a topological $\mathbb{Q}[[h]]$-algebra (see [**CP**]), and acts on certain extension $\hat{V}$ of V, which is a $\mathbb{Q}[[h]]$-module. Here $\mathbb{Z}[q^{\pm 1/d}] \hookrightarrow \mathbb{Q}[[h]]$ via $q^{1/d} = \exp(h/d)$. The fact that V is highest weight implies that J_T acts on $\hat{V}$ by a scalar operator, i.e. there is a scalar $J'_T(V) \in \mathbb{Q}[[h]] \supset \mathbb{Z}[q^{\pm 1/d}]$, such that $J_T = J'_T(V)\,\mathrm{id}$. One can consider $J'_T(V)$ as invariant of the long knot T colored by the highest weight module V.

If V, V' are highest weight $U_q(\mathfrak{g})$-modules and either $V \subset V'$ or V is a quotient of V', then clearly $J'_T(V) = J'_T(V')$.

For every dominant weight λ of $\mathfrak{g}$ there is a finite-dimensional irreducible $U_q(\mathfrak{g})$-module of type 1 V_λ of highest weight λ. For every weight λ, there is a highest weight module W_λ, called the Verma module with highest weight λ. If λ is a dominant weight, then V_λ is a quotient of W_λ. As a result, $J'_T(V_\lambda) = J'_T(W_\lambda)$.

The Weyl group $\mathscr{W}$ of $\mathfrak{g}$ acts on the weight lattice. As usual, we say $\lambda \sim \lambda'$ if $\lambda + \rho$ and $\lambda' + \rho$ are in the same orbit of the Weyl group. Here ρ is the half sum of all positive roots. If λ is dominant and $\lambda' \sim \lambda$, then $W_{\lambda'} \subset W_\lambda$, hence $J'_T(W_\lambda) = J'_T(W_{\lambda'})$.

4. Colored Jones polynomial

In this section we discuss main properties of the colored Jones polynomial, the Melvin-Morton conjecture, the volume conjecture, and the Habiro expansion.

4.1. Quantum link invariants of the Lie algebra $sl_2(\mathbb{C})$. — Like the ring of representations of sl_2, the ring of representations of $U_q(sl_2)$ is simple : for each positive integer n, there is a unique (up to $U_q(sl_2)$-isomorphisms) irreducible n-dimensional $U_q(sl_2)$-module of type 1, denoted by V_n. The 2-dimensional representation V_2 is called the basic representation. For a framed oriented link L with m components, we write

$$J_L(n_1, \dots, n_m) = J_L(V_{n_1}, \dots, V_{n_m}) \in \mathbb{Z}^{\pm 1/4},$$

where the right hand side is the invariant of the framed oriented links whose components are colored by $V_{n_1}, \dots, V_{n_m}$, explained in Subsection 3.6.

In the ring of $U_q(sl_2)$-modules, one has

$$V_n \oplus V_{n-2} \cong V_{n-1} \otimes V_2.$$

Then above identity, the tensor product formula and the additivity described in Subsection 3.6 show that we have the following recurrence relation

$$J_L(\mathbf{k}_1, n, \mathbf{k}_2) = J_{L^{(2)}}(\mathbf{k}_1, n-1, 2, \mathbf{k}_2) - J_L(\mathbf{k}_1, n-2, \mathbf{k}_2), \tag{26}$$

where $L^{(2)}$ is the obtained from L by doubling the component colored by n, using the framing.

The recurrence (26) allows one to calculate the colored Jones polynomial without the theory of quantum group as follows. If all the components a framed oriented link have color 1, then the invariant is 1. If all the colors are 2, the value is the framed Jones polynomial V_L, which can be calculated using the

Kauffman bracket polynomial. If a color is $n \geq 3$, then (26) reduces the calculation of the invariant to the case when the colors are less than n, at the expense of going to more complicated links.

Although the above recurrence allows us to define the colored Jones polynomial without the theory of quantum groups, many properties of the colored Jones polynomial cannot be understood without quantum group theory.

The recurrence relation shows that for a framed oriented knot K, the two sequence of invariants

$$J_K(n), \; n \in \mathbb{Z}_{\geq 1}$$
$$J_{K^{(n-1)}}(2) = V_{K^{(n-1)}}, \; n \in \mathbb{Z}_{\geq 1}$$

are equivalent in the sense that one is expressible in terms of the another by a lower triangular matrix with 1 on the diagonal. The entries of the matrix do not depend on the knots. However, in general, $J_K(n)$ has nicer properties.

4.2. Chebyshev polynomials and negative colors. — The Chebyshev polynomials $T_n(z), S_n(z)$ are defined inductively by

$$T_0 = 2, T_1(z) = z, T_n(z) = zT_{n-1}(z) - T_{n-2}(z)$$
$$S_0 = 1, S_1(z) = z, S_n(z) = zS_{n-1}(z) - S_{n-2}(z).$$

One can also extend the definition of S_n, T_n to $n \in \mathbb{Z}$, using the same recursion formula. Then

$$S_{-1-n} = -S_{-1+n}, \quad T_{-n} = T_n \tag{27}$$
$$T_n = S_n - S_{n-2}.$$

The T_n's are known as Chebyshev's polynomials of type 1, and S_n's are known as Chebyshev's polynomials of type 2.

Exercise 4.1. — Show that if $z = q^{1/2} + q^{-1/2} = [2]$, then

$$T_n(z) = q^{n/2} + q^{-n/2}$$
$$S_{n-1}(z) = [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}.$$

Suppose K is a framed oriented knot. For a non-negative integer n let K^n be the framed knot obtained by replacing the K with n of its parallels. For a polynomial $p(z) = \sum_j a_j z^j \in \mathbb{Z}[z]$ and a framed knot K define

$$V_{p(K)} = \sum_j a_j V_{K^j}.$$

Then the recurrence formula (26) shows that

$$J_K(n) = V_{S_{n-1}(K)}, \tag{28}$$

which can be used as the definition of the colored Jones polynomial. The case of framed links can be done similarly.

Exercise 4.2. — Show that

$$J_U(n) = [n],$$
$$J_H(n, m) = [nm],$$

where U is the unknot and H is the Hopf link.

Equation (28) can be used to define $J_K(n)$ even for $n \leq 0$, since $S_n(z)$ is defined for any $n \in \mathbb{Z}$. From (27) we have

$$J_K(-n) = -J_K(n), \quad J_K(0) = 0.$$

This definition is not artificial as it seems, and can be explained in the framed work of quantum invariants associate to $sl_2(\mathbb{C})$ as follows. Let T be a 1 component string link such that $\hat{T} = K$. Then J_T is a central element of the h-adic version $\hat{U}_h(sl_2)$ of the quantum group. For each integer $n \in \mathbb{Z}$, there is a Verma module W_n whose highest weight is $(n-1)\varpi$, where ϖ is the fundamental weight of sl_2. Then J_T acts on W_n by a scalar operator $J'_T(n)\,\mathrm{id}$, where $J'_T(n) \in \mathbb{Q}[[h]]$, see Subsection 3.6. When $n \geq 1$, V_n is a quotient of W_n. Hence, with $K = \hat{T}$, the closure knot of T,

$$J_K(n) = J'_T \dim_q(V_n) = J'_T(n)[n].$$

The Weyl symmetry says that $W_{-n} \subset W_n$. Hence we have

$$J'_T(-n) = J'_T(n).$$

Thus, although a priori, $J'_T(n) \in \mathbb{Q}[[h]]$, we always have $J'_T(n) \in \mathbb{Z}[q^{\pm 1}]$ if $|n| \neq 0$. (Here $q = \exp(h)$.) If one define $J_{\hat{T}}(n) = J'_T(n)[n]$, then $J_{\hat{T}}(-n) = -J_{\hat{T}}(n)$.

The color $n = 0$ is special. While $J_{\hat{T}}(0) = 0$, the value of $J'_T(0)$ is more complicated. In fact, in general, $J'_T(0) \notin \mathbb{Z}[q^{\pm 1}]$. In [**HL, Ha3**], it was shown that $J'_T(0)$ belongs to the Habiro ring and is equal to the Kashaev invariant, see below.

4.3. Properties of the colored Jones polynomial. — Let K be a framed oriented knot. Since the value of the unknot with color n is $[n] = \frac{q^{n/2} - q^{-n/2}}{q^{n/2} - q^{-n/2}}$, another normalization of of $J_K(n)$ is often used (for $n \neq 0$) :

$$J'_K(n) := \frac{J_K(n)}{[n]}.$$

Thus, if $K = \hat{T}$, where T is a framed 1-component string link, then $J'_K = J'_T$, defined in the previous section.

Proposition 4.3. — *Suppose K, K' are framed oriented knots.*

(1) If K has 0 framing, then $J'_K(n) \in \mathbb{Z}[q^{\pm 1}] = \mathbb{Z}[t^{\pm 4}]$.
(2) $J'_{K!}(n; q) = J'_K(n; q^{-1})$, were $K!$ is the mirror image of K.
(3) $J_K(n; q) = J_{\overleftarrow{K}}(n; q)$, where $\overleftarrow{K}$ is the same knot K with reverse orientation.
(4) $J'_{K\#K'}(n; q) = J'_K(n; q)\, J_{K'}(n; q)$, where $K\#K'$ is the connected sum of K and K'.
(5) If K' is obtained from K by increasing the framing by 1,

$$J_{K'}(n) = q^{(n^2-1)/4} J_K(n).$$

Part (4) is a special case of (25).

Exercise 4.4. — Prove parts (1), (2), (3), and, using (4), prove also (5) of the proposition.

Remark 4.5. — Property (4), showing that J'_K behaves well under connected sum, explains why $V_{S_n(K)}$ is more interesting than V_{K^n}. In the framework of Kauffman's bracket theory, properties (4) and (5) can be proved using the Jones-Wenzl idempotent, see e.g. [**Li**].

Recall that $w(K)$ is the writhe, or the integer value framing, of framed knots. Let

$$\mathring{J}_K(n; q) = q^{-w(K)(n^2-1)/4} J_K(n).$$

From Proposition 4.3(5), $\mathring{J}$ is an invariant of unframed un-oriented knots.

As noted before, the colored Jones polynomial of *framed* oriented links does not actually depend on the orientation of the components. Hence, it is an invariant of framed unoriented links.

4.4. Examples. — In general, it is difficult to have a compact formula of $J_K(n)$, as a function of q and n. Here we give examples of a few known compact formulas. Recall the $J'_K(n) = J_K(n)/[n] \in \mathbb{Z}[q]$.

If K is the right handed trefoil with framing 0, then

$$J'_K(n) = q^{1-n} \sum_{k=0}^{\infty} q^{-kn} (q^{1-n}; q)_k \tag{29}$$

$$= \sum_{k=0}^{\infty} q^{-k(k+3)/2} \prod_{j=1}^{k} (q^n + q^{-n} - q^j - q^{-j}). \tag{30}$$

Formula (29), see [**HL**], is valid when $n > 0$. When $n > 0$, the sum is actually finite and gives a Laurent polynomial, since if $k \geq n$, the k-th summand is 0. Formula (30), see [**Ha1**], is valid for any $0 \neq n \in \mathbb{Z}$, since with $n \neq 0$, the right hand side of (30) is a finite sum (the k-th summand is 0 whenever $k \geq |n|$). An exercise is to show that the two formulas give the same answer when $n > 0$.

When $n < 0$, the right hand side of (29), while not being a finite sum, can be shown to be an element of the *Habiro ring*, and in the Habiro ring, it is equal to the element defined by Formula (30). The Habiro ring [**Ha3**] is defined by

$$\widehat{\mathbb{Z}[q]} = \varprojlim_n \mathbb{Z}[q]/\left((q;q)_n\right), \qquad \text{where } (q;q)_n = \prod_{j=1}^{n} (1-q^j).$$

When $n = 0$, both formulas are infinite sums and equal in the Habiro ring, and give the Kashaev invariant of the knot.

If K is the figure 8 knot with 0 framing (see [**Ha1**])

$$J'_K(n) = \sum_{k=0}^{\infty} \prod_{j=1}^{k} (q^n + q^{-n} - q^j - q^{-j}). \tag{31}$$

See[**Mas**] for a proof of these formulas using skein theory approach.

4.5. Breadth of colored Jones polynomial of adequate links. —

Proposition 4.6 **([Le2])**. — *a) Suppose K is a framed oriented knot with a blackboard diagram D having c crossings. Then for $n \geq 1$,*

$$d_+(J_K(n)) \leq c(n-1)^2 + 2(n-1)s_+(D),$$
$$d_-(J_K(n)) \geq -c(n-1)^2 - 2(n-1)s_-(D).$$

Equalities hold if D is adequate.

b) If K is adequate, then the breadth of $J_K(n)$ grows as a quadratic function in n. If K is a non-trivial alternating knot with c crossings, and $n \geq 1$. Then

$$\mathrm{br}(J_K(n)) = 2c(n-1)^2 + 2(n-1)(c+2).$$

Démonstration. — a) The n-parallel D^n of D will have cn^2 double points. In addition, it is easy to see that $s_\pm(D^n) = ns_\pm(D)$. Hence, Theorem 2.16 says

$$d_+\langle D^n\rangle \leq f(n) := cn^2 + 2n|s_+|.$$

Note that $f(n)$ is a strictly increasing function, $f(n+1) > f(n)$. Recall that $S_n(K) = D^n +$ terms of lower degrees in D. It follows that

$$d_+\langle S_{n-1}(D)\rangle \leq f(n-1) = c(n-1)^2 + 2(n-1)s_+.$$

The proof for d_- is similar.

b) Suppose K is alternating, with a reduced alternating diagram D. Let K' be the framed oriented knot defined by D with blackboard framing. Then K' is the same as K with possibly different framing. Hence, their colored Jones polynomials have the same breadth.

For an alternating diagram one has $s_+ + s_- = c + 2$. By Lemma 2.17 and Theorem 2.16,

$$d_+\langle D^n\rangle = cn^2 + 2ns_+,$$
$$d_-\langle D^n\rangle = -cn^2 - 2ns_-.$$

It follows that $d_+(D^n) > d_+(D^{n-1})$ and $d_-(D^n) < d_-(D^{n-1})$. We have that

$$S_n(K) = D^n + \text{terms of lower degrees in } K,$$

hence $d_\pm(\langle S_n(K)\rangle) = d_\pm(\langle D^n\rangle)$, and

$$\mathrm{br}\, J_K(n) = \mathrm{br}(S_{n-1}(D)) = d_+\langle D^{n-1}\rangle - d_-\langle D^{n-1}\rangle = 2c(n-1)^2 + 2(n-1)(c+2).$$

The case when K is adequate is left as an exercise. □

Exercise 4.7. — Suppose K is an alternating knot with a reduced alternating diagram D which has c_+ positive crossings. Show that c_+ is an invariant of K.

By [**Mur, Tu3**], for an alternating knot, $s_+ - c_+ = \text{signature}(K) + 1$, where signature$(K)$ is the signature of the knot.

4.6. Melvin-Morton conjecture. — We explain here the first important connection between the colored Jones polynomial and the fundamental group.

Fix a 0-framed knot K. The colored Jones polynomial J'_K can be considered as a function on two variables : q and n, where the integer $n \in \mathbb{Z}$ is the color. We will look at various limits of $J'_K(n)$ as $n \to \infty$.

Suppose $u \in \mathbb{C}$ is a complex number, with $e^u = z$, or $u = \log z$. For a fixed z, various values of u satisfying $e^u = z$ differ by a multiple of $2\pi i$.

In this section we consider u near 0. In the next section we consider u near $2\pi i$. In both case, $z = e^u$ is near 1.

For each positive integer n, the function

$$F_{K,n}(u) := J'_K(n, q = \exp(u/n))$$

is an analytic function in $u \in \mathbb{C}$.

Here is the strong Melvin-Morton conjecture, proved in [**GL2**].

Theorem 4.8. — [**GL2**] *For every knot K there is a open set $S_K \subset \mathbb{C}$ containing 0 such that*

$$\lim_{n\to\infty} F_{K,n}(u) = \frac{1}{\nabla_K(e^u)}$$

uniformly on any compact in S_K. Here $\nabla_K(t)$ is the Alexander polynomial of the knot K defined as in Subsection 3.2. In other words, ∇_K is the Alexander polynomial normalized so that $\nabla_K(t) = \nabla_K(t^{-1})$ and $\nabla_K(1) = 1$.

This Alexander polynomial ∇_K is defined in Subsection 3.2.

The original Melvin-Morton conjecture [**MM**] (proved by Bar-Natan and Garoufalidis [**BG**]) says the Maclaurin series of $J'_K(n, q = \exp(u/n))$ converges coefficient-wise to the Maclaurin series of $\frac{1}{\nabla_K(e^u)}$.

The (already proved) Melvin-Morton conjecture provides the first connection between the colored Jones polynomial and the fundamental group, as the Alexander polynomial is an abelian invariant of the fundamental group of knots.

4.7. Volume conjecture. — The volume conjecture [**Kas, MuM**] suggests there is a deep connection between the colored Jones polynomial and the hyperbolic geometry of the knot complement. We will again look at a certain limit of the colored Jones polynomial.

For the volume conjecture, one looks at u near $2\pi i$. The value

$$\mathscr{K}_K(\exp(2\pi i/n)) := F_{K,n}(2\pi i) = J'_K(n; q = \exp(2\pi i/n))$$

is called the Kashaev invariant of K at $q = \exp(2\pi i/n$.

***Conjecture 1* (Volume Conjecture).** — *For any knot K,*

$$\lim_{n\to\infty} \frac{\log |J'_K(n; q = \exp(2\pi i/n))|}{n} = \frac{\text{Vol}(K)}{2\pi}.$$

Here Vol(K) is the hyperbolic volume of the knot complement, defined as in Subsection 3.2.

So far the volume conjecture has been verified only for a few hyperbolic knots. For more on the volume conjecture, see [**Kas, MuM, Muk**].

4.8. Habiro's expansion of the colored Jones polynomial. — Habiro [**Ha3**], using quantum group theory, showed that for every zero-framed knot K and non-negative integer k, there exists $C_K(m) \in \mathbb{Z}[q^{\pm 1}]$ such that

$$J'_K(n) = \sum_{m=0}^{\infty} C_K(m) \prod_{j=1}^{m} (q^n + q^{-n} - q^j - q^{-j}). \tag{32}$$

The existence and uniqueness of $C_K(m) \in \mathbb{Q}(q)$ satisfying (32) is easy to prove. The real content of Habiro's result is that $C_K(m)$ are Laurent polynomials with integer coefficients, and formula expressing $C_K(m)$ in terms of $J_K(n)$. So far, there is no known proof of Habiro's result using Kauffman's bracket theory.

The existence of $C_K(k)$ and (32) have found applications in many works, see [**Ha3, GL2**]. We will have one application in the next section.

When $n = 0$, the right hand side of (32) is an element of the Habiro ring, and hence can be evaluated at any root of unity. It is an easy exercise to show that $\mathscr{K}_K(\exp(2\pi i/n)) := J'_K(n; q = \exp(2\pi i/n))$ is equal to $J'_K(0; q = \exp(2\pi i/n))$. Thus, the Kashaev invariant of K can be conveniently described by the function $J'_K(0)$, which can be evaluated when q is a root of unity.

4.9. Colored Jones polynomial at roots of 1. — In the volume conjecture, we look at the value of $J'_K(n; q)$ at q a root of unity.

The colored Jones polynomial enjoy the following symmetry.

Proposition 4.9. — *For every zero-framed knot K and every root ξ of unity of order r,*

$$J'_K(n) = J'_K(r - n) = J'_K(r + n),$$

when evaluated at $q = \xi$.

This type of symmetry was first discovered by Kirby and Melvin [**KM**] for sl_2 quantum invariants. For general Lie algebra see [**Le1**] and references therein.

Exercise 4.10. — Use expression (32) to prove Proposition 4.9.

For example, the stronger volume conjecture

$$\lim_{n,m\to\infty, n/m\to 1} \frac{\log |J'_K(n; q = \exp(2\pi i/m))|}{n} = \frac{\mathrm{Vol}(K)}{2\pi},$$

where the single limit is replace by a double limit, is wrong. This is because if $m = n + 1$, then

$$J'_K(n; q = \exp(2\pi i/(n + 1))) = J'_K(1; q = \exp(2\pi i/(n + 1))) = 1,$$

and the limit would be 0.

This is very different from the case when u is near 0, considered in the strong Melvin-Morton conjecture. There, due to the uniform convergence, similar double limits exist.

5. Recurrence relation : holonomicity

In this section we sketch a proof of the fact that for every knot K, the sequence of Jones polynomials $J_K(n), n \in \mathbb{Z}$, satisfies a recurrence relation. Results of this section are taken from [**GL1, Le2**].

We will use the notation $\mathcal{R} = \mathbb{C}[t]$, and $q = t^4$. Also $\mathbb{N}$ is the set of non-negative integers.

5.1. Recurrence relation with polynomial coefficients. — The Fibonacci sequence is an example of a sequence satisfying a recurrence relation with constant coefficients :

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

We will need two initial values in order to determine F_n. For example, $F_1 = 1, F_2 = 1$.

And here is an example of a recurrence relation with polynomial coefficients :

$$(n-4)F_n = (n^3 - 4n^2 - 1)F_{n-1} + (n^2 - 3n - 2)F_{n-2}.$$

In general, two initial values are not enough, but 4 initial values, $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = \sqrt{2}$, are enough to determine the whole sequence.

5.2. q-holonomicity, one variable. — We will consider q-analogs of recurrence relations with polynomial coefficients.

Suppose V is a $\mathbb{Z}[t^{\pm 1}]$-modude. The set $\mathrm{Map}(\mathbb{Z}, V)$ of all functions from $\mathbb{Z}$ to V is also a $\mathbb{Z}[t^{\pm 1}]$-module.

There are two operators L, M acting on $\mathrm{Map}(\mathbb{Z}, V)$:

$$(Lf)(n) := f(n+1)$$
$$(Mf)(n) := t^{2n} f(n).$$

Their inverses L^{-1}, M^{-1} exist. One has $LM = t^2 ML$. The algebra

$$\mathcal{T} := \mathbb{Z}[t^{\pm 1}]\langle L^{\pm 1}, M^{\pm 1}\rangle / (LM = t^2 ML)$$

is called the *quantum torus*. We have seen that $\mathcal{T}$ acts on $\mathrm{Map}(\mathbb{Z}, V)$, or $\mathrm{Map}(\mathbb{Z}, V)$ is a left $\mathcal{T}$-module.

Let $\mathcal{T}_+ \subset \mathcal{T}$ be the $\mathbb{Z}[t^{\pm 1}]$-submodule spanned by all monomials of the form $L^k M^p$ with $k, p \geq 0$. Then $\mathcal{T}_+$ is known as the quantum plane. It is easy to see that the set $\{M^a L^b \mid a, b \in \mathbb{Z}\}$ is a $\mathbb{Z}[t^{\pm 1}]$-basis of $\mathcal{T}$. Similarly, the set $\{M^a L^b \mid a, b \in \mathbb{N}\}$ is a $\mathbb{Z}[t^{\pm 1}]$-basis of $\mathcal{T}_+$.

Definition 2. — *A function* $f \in \mathrm{Map}(\mathbb{Z}, V)$ *is called* q-holonomic *if there is* $0 \neq \alpha \in \mathcal{T}$ *such that* $\alpha(f) = 0$. *Similarly, a function* $f \in \mathrm{Map}(\mathbb{N}, V)$ *is q-holonomic if there is* $0 \neq \alpha \in \mathcal{T}_+$ *such that* $\alpha(f) = 0$.

In general, the set $\mathcal{A}_f := \{\alpha \in \mathcal{T} \mid \alpha(f) = 0\}$ is called the *annihilator ideal* of f, which is a left ideal of $\mathcal{T}$. Thus, f is q-holonomic if and only its annihilator ideal is not 0.

We will show that for every knot K, the function $J_K : \mathbb{Z} \to \mathcal{R} = \mathbb{C}[t^{\pm 1}]$ is q-holonomic.

An important property of q-holomorphic function is the following. If f is q-holonomic, then f is totally determined by a finite set of initial values and a non-zero recurrence relation $\alpha \in \mathcal{A}_f$. This means, there exists $n, m \in \mathbb{Z}$, depending on α, such that if $\alpha(f) = \alpha(g) = 0$ and $f(j) = g(j)$ for $n \leq j \leq m$, then $f = g$.

The set of possible images of a fixed f under $\mathcal{T}$ is $\mathcal{T} \cdot f = \mathcal{T}/\mathcal{A}_f$. Hence, if f is not q-holomorphic, then $\mathcal{T} \cdot f \cong \mathcal{T}$ is much bigger than $\mathcal{T} \cdot g = \mathcal{T}/\mathcal{A}_g$ for some q-holonomic g.

Exercise 5.1. — Show that each of functions $n \to t^{2n}$ and $n \to t^{4n^2}$ is q-holonomic. However, $n \to t^{8n^3}$ is not q-holonomic.

Show that the function

$$H(n) = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

is q-homonomic.

5.3. q-holonomicity, many variables. — To show that $J_K : \mathbb{Z} \to \mathcal{R} = \mathbb{C}[t^{\pm 1}]$ is q-holonomic, we decompose J_K into pieces, show that each piece is q-holonomic, and the way we assemble pieces together preserves q-holonomicity. For this purpose, we need q-holonomicity of many variables.

For a function $f : \mathbb{Z}^r \to V$, with $r \geq 2$ the definition of q-holonomicity is more complicated. The function must satisfy sufficiently many recurrence relations in order to be q-holonomic. But how many recurrence relations would be enough ?

Let

$$\mathcal{T}_r = \mathbb{Z}[t^{\pm 1}]\langle L_1^{\pm 1}, \dots, L_r^{\pm 1}, L_1^{\pm 1}, \dots, L_r^{\pm 1}\rangle/(L_i M_i = t^2 M_i L_i, L_i M_j = M_j L_i \text{ for } \ i \neq j).$$

The algebra $\mathcal{T}_r$ acts on $\mathrm{Map}(\mathbb{Z}^r, V)$, where V is any $\mathbb{Z}[t^{\pm 1}]$-module, by

$$(L_i f)(n_1, \dots, n_i, \dots, n_r) = f(n_1, \dots, n_i + 1, \dots, n_r)$$
$$(M_i f)(n_1, \dots, n_r) = t^{2n_i} f(n_1, \dots, n_r).$$

Let $\mathcal{T}_{r,+}$ be the subalgebra of $\mathcal{T}_r$ generated by non-negative powers of M_j, L_j.

Suppose $f \neq 0$. From f, by actions of $\mathcal{T}_{r,+}$, we get other functions, all of which are $(\mathcal{T}_{r,+}) \cdot f$. In general, the more recurrence relations f satisfies, the smaller $(\mathcal{T}_{r,+}) \cdot f$ is.

Informally, f is q-holonomic if the $(\mathcal{T}_{r,+}) \cdot f$ is as small as possible, in some complexity measure. A precise definition is the following. Berstein's inequality tells us that the *dimension* of $(\mathcal{T}_{r,+}) \cdot f$ is always $\geq r$, and one says f is *q-holonomic* if $f = 0$ or the dimension of $(\mathcal{T}_{r,+}) \cdot f$ is exactly r.

The dimension of $(\mathcal{T}_{r,+}) \cdot f$ can be defined as follows. Let $(\mathcal{T}_{r,+})^{\leq N}$ be the $\mathcal{R}$-span of all monomials in M_j, L_k with total degree $\leq N$. An analog of Hilbert's theorem for this non-commutative setting holds true : The $\mathbb{C}(t)$-dimension of $(\mathcal{T}_{r,+})^{\leq N} \cdot f$ is a polynomial in N, for big enough N. The degree of this polynomial is called the dimension of $\mathcal{T}_+ \cdot f$.

Another way to define the dimension : Suppose W is a non-zero $\mathcal{T}_r$-module. Its co-dimension and dimension are defined by

$$\mathrm{codim}(W) = \min\{j \in \mathbb{N} \mid \mathrm{Ext}^j_{\mathcal{T}_r}(W, \mathcal{T}_r) \neq 0\}, \quad \dim(N) = 2r - \mathrm{codim}(W).$$

Then Berstein inequality (proved by Sabbah [**Sab**] in the q-case) says that if $W \neq 0$ is finitely generated, then $\dim(W) \geq r$. *A $\mathcal{T}_r$-module W is q-holonomic* if either $W = 0$ or $\dim(W) = r$. A function $f \in \mathrm{Map}(\mathbb{Z}^r, V)$ is *q-hononomic* if the module $\mathcal{T}_r \cdot f$ is q-holonomic.

Exercise 5.2. — Show that when $r = 1$, this definition of q-holonomicity is equivalent to the one given in Subsection 5.2.

One can define in a similar fashion *q-holomorphic functions with domain $\mathbb{N}^n$*, using only non-negative powers of L_j, M_k.

5.4. Examples of q–holonomic functions. — Here are a few examples of q–holonomic functions. In fact, we will encounter only sums, products, extensions, specializations, diagonals, and multisums of these functions. We use $q = t^4$.

Recall that for $n \in \mathbb{N}$,

$$(x; q)_n := \prod_{j=1}^{n} (1 - x q^{j-1}).$$

For $n, k \in \mathbb{Z}$, let

$$F(n,k) := \begin{cases} (q^n; q^{-1})_k, & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$G(n,k) := \begin{cases} \frac{(q^n; q^{-1})_k}{(q^k; q^{-1})_k} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}.$$

Then both f and g, as well as the delta function $\delta_{n,k}$, are q–holonomic. Note that $g(n,k)$ is the q-binomial coefficient and $f(n,k)$ is the q-combination number.

Exercise 5.3. — Prove that F, G, and $\delta_{n,k}$ are q-holonomic.

5.5. Properties of q-holonomic functions. —

- Sums and products of q–holonomic functions are q–holonomic.
- Specializations and extensions of q–holonomic functions are q–holonomic. In other words, if $f(n_1,\dots,n_m)$ is q–holonomic, then so are the functions
$$g(n_2,\dots,n_m) := f(a,n_2,\dots,n_m) \quad \text{for fixed } a$$
$$\text{and} \qquad h(n_1,\dots,n_m,n_{m+1}) := f(n_1,\dots,n_m).$$
- Linear substitution. If $f(n_1,\dots,n_m)$ is q–holonomic, then so is the function, $g(n'_1,\dots,n'_{m'})$, where each n'_j is a $\mathbb{Z}$-linear function of n_i.
- In particular, diagonals of q–holonomic functions are q–holonomic. That is, if $f(n_1,\dots,n_m)$ is q–holonomic, then so is the function
$$g(n_2,\dots,n_m) := f(n_2,n_2,n_3,\dots,n_m).$$
- Multisums of q–holonomic functions are q–holonomic. In other words, if $f(n_1,\dots,n_m)$ is q–holonomic, the so are the functions g and h, defined by
$$g(a,b,n_2,\dots,n_m) := \sum_{n_1=a}^{b} f(n_1,n_2,\dots,n_m)$$
$$h(a,n_2,\dots,n_m) := \sum_{n_1=a}^{\infty} f(n_1,n_2,\dots,n_m)$$
(assuming that the latter sum is finite for each a).

5.6. The colored Jones polynomial is q-holonomic. — The quantum group definition of the colored Jones polynomial, discussed in Subsection 3.6 leads to the following formula for the colored Jones of a knot.

Let V_n be the n-dimensional irreducible $U_q(sl_2)$ used to define the colored Jones polynomial with color n. We have the YB operator $R : V_n \otimes V_n \to V_n \otimes V_n$ and $\eta : V_n \to V_n$.

Explicit formulas for R and η are known. There is an ordered basis $\{e_0, e_1, \dots, e_n\}$ of V_{n+1} such that
$$R^{\pm 1}(e_a \otimes e_b) = \sum R^{\pm 1}(n;a,b,c,d)\, e_c \otimes e_d, \quad \eta(e_a) = \eta(n;a)e_a,$$
where

(33) $$R(n;a,bc,d) = q^{(n+nd+nb-ab-dc)/2}\, F(c,c-b)G(n-a,d-a)\delta_{a+b,c+d}$$

(34) $$R^{\pm 1}(n;a,bc,d) = (-1)^{b-c} q^{(-n-nb-nd+bd+ac-b+c)/2}\, F(a,a-d)G(n-c,b-c)\delta_{a+b,c+d}$$

(35) $$\eta(n;a) = q^{(2a-n)/2}.$$

Here F, G are functions given in Subsection 5.4.

A basis of $(V_{n+1})^{\otimes k}$ is $\{e(\mathbf{m}) \mid \mathbf{m} \in \{0,1,\dots,n\}^k$, where
$$e(\mathbf{m}) = e_{m_1} \otimes e_{m_2} \otimes \cdots \otimes e_{m_k} \quad \text{for } \mathbf{m} = (m_1,\dots,m_k) \in \{0,1,\dots,n\}^k\}.$$
Using this basis, any linear operator $\Omega : (V_{n+1})^{\otimes k} \to (V_{n+1})^{\otimes k}$ is describes by its matrix elements $\Omega(n;\mathbf{m};\mathbf{m}') \in \mathbb{Q}(q^{1/d})$, where
$$\Omega(e(\mathbf{m})) = \sum_{\mathbf{m}' \in \{0,\dots,n\}^k} \Omega(n;\mathbf{m},\mathbf{m}').$$

Fix a positive integer k. Suppose for each non-negative integer n we have a $\mathbb{Q}(q^{1/d})$-linear operator $\Omega : (V_{n+1})^{\otimes k} \to (V_{n+1})^{\otimes k}$. We say that Ω is *q-holonomic* (in the chosen basis) if the matrix element $\Omega(n;\mathbf{m};\mathbf{m}')$ is q-holonomic in all variables, i.e. there is a q-holonomic function $F : \mathbb{N}^{2k+1} \to \mathbb{Q}(q)$ such that $\Omega(n;\mathbf{m};\mathbf{m}') = F(n,\mathbf{m},\mathbf{m}')$ whenever $\mathbf{m},\mathbf{m}' \in \{0,\dots,n\}^k$.

From the properties of q-holomorphic functions listed in Subsection 5.5, one can easily show that if $\Omega_1, \Omega_2 : (V_{n+1})^{\otimes k} \to (V_{n+1})^{\otimes k}$ are q-holonomic, then $\Omega_1\Omega_2$ is also q-holonomic. Also, if $\Omega_1 : (V_{n+1})^{\otimes k_1} \to (V_{n+1})^{\otimes k_1}$ and $\Omega_2 : (V_{n+1})^{\otimes k_2} \to (V_{n+1})^{\otimes k_2}$ are q-holonomic, then $\Omega_1 \otimes \Omega_2$ is also q-holonomic.

The right hand side of (33) can be defined for any $(n, a, b, c, d) \in \mathbb{Z}^5$, and is q-holomorphic functions from $\mathbb{Z}^5$ to $\mathbb{Z}[q^{\pm 1}]$, see Subsection 5.5. In other words, $R : (V_{n+1})^{\otimes 2} \to (V_{n+1})^{\otimes 2}$ is q-holonomic. Similarly, (34) and (35) show that R^{-1} and η are q-holonomic.

It follows that $\rho(\beta)$ is q-holonomic for every braid $\beta \in \mathfrak{B}_k$. Similarly, $\eta^{\otimes k}$ is q-holonomic. Because taking trace preserves the q-holonomicity (again by properties listed in Subsection 5.5),

$$J_K(n+1) = \mathrm{tr}_q^{V_{n+1}}(\rho(\beta)) = \mathrm{tr}(\rho(\beta)\,\eta^{\otimes k}, (V_{n+1})^{\otimes k})$$

is q-holonomic in $n \in \mathbb{N}$.

With a little more effort one can extend the statement to include negative colors, and get the following.

Theorem 5.4. — *For every framed oriented knot K, the function $J_K : \mathbb{Z}^n \to \mathbb{Z}[t^{\pm 1}]$ is q-holonomic.*

Let $\mathcal{A}_K$ be the recurrence ideal of $J_K : \mathbb{Z} \to \mathcal{R}$, which is non-zero by Theorem 5.4.

5.7. An example. — For the right-handed trefoil with 0 framing, one has

$$J_K(n) = \frac{t^{2-2n}}{1-t^{-4}} \sum_{k=0}^{n-1} t^{-4nk} \prod_{i=0}^{k} (1 - t^{4i-4n}).$$

The function J_K satisfies $\alpha\, J_K = 0$, where

$$\alpha = (t^4 M^{10} - M^6)L^2 - (t^2 M^{10} + t^{-18} - t^{-10} M^6 - t^{-14} M^4)L + (t^{-16} - t^{-4} M^4). \tag{36}$$

Together with the initial conditions $J_K(0) = 0, J_K(1) = 1$, this recurrence relation determines $J_K(n)$ uniquely.

5.8. Generator of the recurrence ideal. — For a field $\mathcal{F}$, the ring $\mathcal{F}[x^{\pm 1}]$ of Laurent polynomials in x with coefficients in $\mathcal{F}$ is a principal ideal domain (PID), but the ring $\mathcal{F}[x^{\pm 1}, y^{\pm 1}]$ is not a PID. For this reason, $\mathcal{T}$ is not a left PID, i.e. not every left ideal of $\mathcal{T}$ is generated by a single element, since $\mathcal{T}/(1+t) = \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$. Note, however, if we adjoin to $\mathcal{F}[x^{\pm 1}, y^{\pm 1}]$ the inverses of all non-zero elements in $\mathcal{F}[x^{\pm 1}]$, then we get a PID, since the new ring is $\mathcal{F}(x)[y^{\pm 1}]$.

The annihilator ideal α_K is a left ideal of $\mathcal{T}$. In general, α_K is not generated by one element. Even for the trivial knot, the recurrence ideal α_K is not principal.

Garoufalidis [**Ga2**] noticed that by adding to $\mathcal{T}$ all the inverses of non-zero polynomials in M one gets a principal ideal domain $\tilde{\mathcal{T}}$. Formally $\tilde{\mathcal{T}}$ can be defined as follows. Let $\mathcal{R}(M)$ be the fractional field of the polynomial ring $\mathcal{R}[M]$. Let $\tilde{\mathcal{T}}$ be the set of all Laurent polynomials in the variable L with coefficients in $\mathcal{R}(M)$:

$$\tilde{\mathcal{T}} = \{\sum_{k\in\mathbb{Z}} a_k(M) L^k \mid \quad a_k(M) \in \mathcal{R}(M), \;\; a_k = 0 \quad \text{almost everywhere}\},$$

and define the product in $\tilde{\mathcal{T}}$ by $a(M)L^k \cdot b(M)L^l = a(M)\, b(t^{2k}M) L^{k+l}$.

Then it is known that every left ideal in $\tilde{\mathcal{T}}$ is principal, and $\mathcal{T}$ embeds as a subring of $\tilde{\mathcal{T}}$. The extension $\tilde{\mathcal{A}}_K := \tilde{\mathcal{T}}\mathcal{A}_K$ of $\mathcal{A}_K$ in $\tilde{\mathcal{T}}$ is then generated by a single polynomial

$$\alpha_K(t; M, L) = \sum_{i=0}^{d} \alpha_{K,i}(t; M) L^i \;\in\; \mathcal{T}_+,$$

where the degree in L is assumed to be minimal and all the coefficients $\alpha_{K,i}(t; M) \in \mathbb{Z}[t^{\pm 1}, M]$ are assumed to be co-prime. That α_K can be chosen to have integer coefficients follows from the fact that $J_K(n) \in \mathbb{Z}[t^{\pm 1}]$. One can easily shows that $\alpha_K(t; M, L)$ annihilates $J_K(n)$, except for a finite number of values of n. Note that $\alpha_K(t; M, L)$ is defined up to a factor $\pm t^a M^b, a, b \in \mathbb{Z}$. We will call α_K the *recurrence polynomial* of K. For example, the polynomial α in the previous subsection is the recurrence polynomial of the right-handed trefoil.

Lemma 5.5. — *Suppose K' is obtained from K by increasing the framing by $k \in \mathbb{Z}$, and $\alpha_K = \sum_{j=0}^{d} a_j(M,t)L^j$ is the recurrence polynomial for K. Then*

$$\alpha'_K := \sum_{j=0}^{d} (-t)^{-kj^2} M^{-jk} a_j(M,t)$$

is the recurrence polynomial for K'.

Démonstration. — The recurrence polynomial of a knot can be characterize as the polynomial in $\tilde{\mathcal{A}}_K \cap \mathcal{T}_+$ with smallest L-degree.

By Proposition 4.3,

$$J_{K'}(n) = (-t)^{k(n^2-1)} J_K(n).$$

From here one can easily show that $\alpha'_K \in \tilde{\mathcal{A}}_{K'}$. The map $\alpha_K \to \alpha'_K$ preserves the L-degree, and is invertible. Hence, α'_K must be the recurrence polynomial of K'. □

By virtue of the lemma, we only need to investigate α_K with K having framing 0.

Proposition 5.6. — *Suppose K is a 0-framed knot. Then α_K can be chosen so that it has only even powers in t and even powers in M, i.e.*

$$\alpha_K \in \mathcal{T}^{\mathrm{ev}} := \mathbb{Z}[t^{\pm 2}]\langle M^{\pm 2}, L^{\pm 1}\rangle/(LM^2 = t^4 M^2 L).$$

Sketch of proof. — If K has 0-framing, then $J_K(n) \in \mathbb{Z}[t^{\pm 2}]$. Hence, one can choose α_K so that it contains only even powers of t.

One has $J_K(n) = [n]J'_K(n)$ and $J'_K(n) \in \mathbb{Z}[t^{\pm 4}]$, see Proposition 4.3. Besides, $[n] = (t^{2n} - t^{-2n})/(t^2 - t^{-2}) \in t^{2(n-1)}\mathbb{Z}[t^{\pm 4}]$. From here one can easily show that α_K can be chosen so that $\alpha_K \in \mathcal{T}^{\mathrm{ev}}$. □

From now on we will assume K has framing 0, and $\alpha_K \in \mathcal{T}^{\mathrm{ev}}$.

5.9. Effect of Weyl symmetry. — Let $\varphi : \mathcal{T} \to \mathcal{T}$ be the $\mathcal{R}$-algebra involution defined by

$$\varphi(M^a L^b) = M^{-a} L^{-b}.$$

(Check that φ is a well-defined algebra involution of $\mathcal{T}$!)

Proposition 5.7. — *The annihilator ideal $\mathcal{A}_K$ is invariant under φ.*

Exercise 5.8. — Prove the proposition, using the fact that $J_K(-n) = -J_K(n)$.

The symmetry of α_K under the involution φ is important for us. Since $\mathcal{A}_K$ is invariant under φ, we want to understand if one can find a generator of $\tilde{\mathcal{A}}_K$ which is an eigenvector of the involution φ. In general, this does not hold true. However, if one adjoint $L^{1/2}$ to $\mathcal{T}$, then such a generator exists. Recall that α_K is defined up to $\pm t^a M^b, a, b \in \mathbb{Z}$.

Proposition 5.9. — *Suppose α_K has L-degree d. There is a normalization of α_K by $\pm t^a M^b, a, b \in \mathbb{Z}$ such that $L^{-d/2}\alpha_K$ is φ-symmetric or φ-anti-symmetric, i.e.*

$$\varphi(L^{-d/2}\alpha_K) = \pm L^{-d/2}\alpha_K.$$

Here we adjoint $L^{\pm 1/2}$ to $\mathcal{T}$, such that $L^{1/2}M = tML^{1/2}$.

We don't need the proposition in the future, and leave the proof as an exercise for the reader.

5.10. Degree 1 recurrence relation. — It turns out that if J_K has a recurrence relation of degree 1, then the breadth of $J_K(n)$ can grow at most linearly with n. In the following statement we use the Weyl symmetry in an essential way.

Proposition 5.10. — *Suppose the annihilator polynomial α_K has L-degree 1, where K is a 0-framed knot. Then there is a constant C such that for any $n \geq 1$,*

$$\mathrm{br}(J_K(n)) \leq Cn.$$

Consequently, if K is an alternating non-trivial knot, or K is a non-trivial adequate knot, then α_K has L-degree ≥ 2.

Sketch of Proof. — Assume $\alpha_K = P(t;M)\,L + P_0(t;M)$, where $P, P_0 \in \mathbb{Z}[t^{\pm 2}, M^{\pm 2}]$. Since $\varphi(\alpha_K) = P(t;M^{-1})\,L^{-1} + P_0(t;M^{-1})$ is also in the recurrence ideal, it is divisible by α_K.

One can show that, after normalizing both P, P_0 by a same power of M, one has

$$P_0(t;M) = \pm P(t;t^{-2}M^{-1}).$$

The equation $\alpha_K J_K = 0$ can now be rewritten as

$$J_K(n+1) = \mp \frac{P(t;t^{-2-2n})}{P(t;t^{2n})} J_K(n).$$

Thus

$$\mathrm{br}(J_K(n+1)) = \mathrm{br}(J_K(n)) + \mathrm{br}(P(t;t^{-2-2n})) - \mathrm{br}(P(t;t^{2n})).$$

It is easy to see that for n big enough, the difference of the breadths $\mathrm{br}(P(t;t^{-2-2n})) - \mathrm{br}(P(t;t^{2n}))$ is a constant depending only on the polynomial $P(t;M)$, but not on n. From the above equation it follows that the breadth of $J_K(n)$, for n big enough, is a linear function on n. □

Corollary 5.11. — *Suppose K is an adequate knot. Then recurrence polynomial α_K has L-degree at least 2.*

Démonstration. — Since the breadth of $J_K(n)$ has quadratic growth by Proposition 4.6, the corollary follows from Proposition 5.10. □

5.11. Linear factor $L-1$. — In the following statement, we use the Melvin-Morton conjecture, see Subsection 4.6.

Proposition 5.12. — *Suppose K is a 0-framed knot. At $t = \pm 1$, the recurrence polynomial α_K is divisible by $L-1$. In other words, $\dfrac{\alpha_K|_{t=-1}}{L-1} \in \mathbb{Z}[M,L]$.*

Sketch of Proof. — Suppose $\alpha_K = \sum_{j=0}^d a_j(t,M)L^j$. One has

$$\sum_{j=0}^d a_j(t,t^{2n}) J_K(n+j) = 0.$$

Since $J_K(n) = [n]J'_K(n)$, we have

$$\sum_{j=0}^d a_j(t,t^{2n})\,(t^{2n+2j} - t^{-2n-2j})\,J'_K(n+j) = 0.$$

Setting $t = e^{u/4n}$ for small enough $|u|$ and taking the limit as $n \to \infty$, using Theorem 4.8, we have

$$\sum_{j=0}^d a_j(1,e^{u/2})\,(e^{u/2} - e^{-u/2}) \frac{1}{\nabla(e^{u/2})} = 0.$$

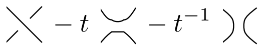
It follows that $\sum_{j=0}^d a_j(1,e^{u/2}) = 0$ for small $|u|$. Hence, $\alpha_K|_{t^2=1,M=z,L=1} = 0$, which is equivalent to the lemma. □

6. Kauffman bracket skein modules

In this section we discuss the Kauffman bracket skein module which was introduced by Przytycki [**Pr**] and Turaev [**Tu2**]. Bracket skein modules have closed connections to character varieties and the colored Jones polynomial, and will serve as a bridge between the A-polynomial and colored Jones polynomial, as first noticed in [**FGL**].

Recall that we use $\mathcal{R} = \mathbb{C}[t^{\pm 1}]$.

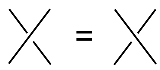
6.1. Skein modules. — A *framed link* in an oriented 3-manifold Y is a disjoint union of embedded circles, equipped with a non-zero normal vector field. We will considered here only unoriented framed links. Framed links are considered up to isotopy. In all figures we will draw framed links, or part of them, by lines as usual, with the convention that the framing is blackboard. Let $\mathcal{L}$ be the free $\mathcal{R}$-module with basis the set of isotopy classes of framed links in the manifold Y, including the empty link. Let Rel be the smallest submodule containing all expressions of the form

(37) 

(38) $$\bigcirc + (t^2 + t^{-2})\emptyset,$$

where the links in each expression are identical except in a small ball in which they look like depicted. The quotient $\mathcal{S}(Y) := \mathcal{L}/\mathrm{Rel}$ is called the Kauffman bracket skein module, or just skein module, of Y.

Remark 6.1. — The fact that Y is oriented is very important. This is because if we do a reflection in the plane of the page, the last two terms of (37) do not change, but the first term switches its over/under crossing. The reflection changes the orientation of the small ball mentioned above and would be prohibited if an orientation is fixed. Without orientation, in the skein module one would have



In general, there is no natural algebra structure on $\mathcal{S}(Y)$. There are, however, two important cases when $\mathcal{S}(Y)$ has a natural algebra structure. We describe here the first case, and will discuss the second one later in Subsection 6.8.

When $Y = \Sigma \times [0,1]$, the cylinder over an oriented surface Σ, we also use the notation $\mathcal{S}(\Sigma)$ for $\mathcal{S}(Y)$. In this case $\mathcal{S}(\Sigma)$ has an algebra structure induced by the operation of gluing one cylinder on top of the other, i.e. $\ell\ell'$ is the result of placing ℓ atop ℓ'. If $K \subset \Sigma$ is an embedded closed curve, i.e. a knot in Σ, then K^n is the n copies of parallel of K. In general, $\mathcal{S}(\Sigma)$ is not commutative.

More generally, if Σ is a part of the boundary of an oriented 3-manifold, then the operation of gluing the cylinder over Σ to Y induces a left $\mathcal{S}(\Sigma)$-module structure on $\mathcal{S}(M)$.

It may happen that $\Sigma_1 \not\cong \Sigma_2$ but $\Sigma_1 \times [0,1] \cong \Sigma_2 \times [0,1]$. Then $\mathcal{S}(\Sigma_1) \cong \mathcal{S}(\Sigma_2)$ as $\mathcal{R}$-modules, but in general, $\mathcal{S}(\Sigma) \not\cong \mathcal{S}(\Sigma_2)$ as $\mathcal{R}$-algebras.

We have the following description of a basis of $\mathcal{S}(\Sigma$.

Proposition 6.2 ([**CP**]). — *Suppose Σ is an oriented surface. As an $\mathcal{R}$-module, $\mathcal{S}(\Sigma)$ is free with basis the set of all isotopy classes of links* embedded in Σ *without trivial components.*

Exercise 6.3. — Suppose $f : Y_1 \hookrightarrow Y_2$ is an embedding. Show that $L \to f(L)$ gives rise to a well-define $\mathcal{R}$-module map $f_* : \mathcal{S}(Y_1) \to \mathcal{S}(Y_2)$.

Similarly, if $f : \Sigma \hookrightarrow T$ is an embedding of an oriented surface Σ into an oriented 3-manifold, then $L \to f(L)$ gives rise to a well-define $\mathcal{R}$-module map $f_* : \mathcal{S}(\Sigma) \to \mathcal{S}(Y)$.

6.2. Important examples. —

Example 6.4 (**$Y = \mathbb{R}^3, S^3$, or 3-ball**). — When Y is the 3-space $\mathbb{R}^3$ or the 3-sphere S^3, the skein module $\mathcal{S}(Y)$ is free over $\mathcal{R}$ of rank one, and is spanned by the empty link. As an exercise, show that if ℓ is a framed link in $\mathbb{R}^3$, then its value in the skein module $\mathcal{S}(\mathbb{R}^3)$ is $\langle \ell \rangle$ times the empty link, where $\langle \ell \rangle \in \mathcal{R}$ is the Kauffman bracket of ℓ.

From this result, one could think of $\mathcal{S}(Y)$ as the space of all Kauffman bracket type polynomial of framed links in Y.

If Σ is a 2-dimensional disk, then $\Sigma \times [0,1]$ is 3-dimensional disk, and then $\mathcal{S}(\Sigma) \cong \mathcal{R}$ as $\mathcal{R}$-algebras.

***Example 6.5* (The annulus).** — Let $\mathbb{A}$ be an annulus. Let z be the core of $\mathbb{A}$, i.e. the only non-trivial loop embedded in $\mathbb{A}$. From Proposition 6.2, $\mathcal{S}(\mathbb{A})$ is the free $\mathcal{R}$-modules with basis $\{z^n, n \in \mathbb{N}\}$. It follows that as an $\mathcal{R}$-algebra, $\mathcal{S}(\mathbb{A}) = \mathcal{R}[z]$.

Instead of the $\mathcal{R}$-basis $\{1, z, z^2, \dots\}$, two other bases are often useful. Namely, each of $\{T_n(z) \mid n \in \mathbb{N}\}$ and $\{S_n(z) \mid n \in \mathbb{N}\}$ is a $\mathcal{R}$-basis of $\mathcal{S}(ST)$. Here T_n, S_n are the Chebyshev polynomials defined in section 4.2.

A framed knot K in $\mathbb{R}^3$ gives rise to an embedding $f : \mathbb{A} \hookrightarrow \mathbb{R}^3$, which is defined up to isotopy. Then colored Jones polynomial is

$$J_K(n) = (-1)^{n-1} f_*(S_{n-1}(z)), \quad n \in \mathbb{Z}.$$

***Example 6.6* (The torus).** — Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the torus with a fixed pair (μ, λ) of oriented simple closed curves intersecting at exactly one point. For co-prime integers k and l, let $\lambda_{k,l}$ be an simple closed curve on the torus homologically equal to $k\mu + l\lambda$. We will consider $\lambda_{k,l}$ as an *unoriented* curve. Then $\lambda_{k,l} = \lambda_{-k,-l}$. Proposition 6.2 shows that $\mathcal{S}(\mathbb{T}^2)$ is the free $\mathcal{R}$-module with basis the set of all $\lambda_{k,l}$ and their parallels, and the trivial link.

Bullock and Przytycki [**BP**] showed that $\mathcal{S}(\mathbb{T}^2)$ is generated over $\mathcal{R}$ by 3 elements μ, λ, and $\lambda_{1,1}$, subject to some explicit relations. If one adds the inverse of $(1 - t^4)$ to the ground ring $\mathcal{R}$, then $\mathcal{S}(\mathbb{T}^2)$ is generated by just two elements μ, λ.

Let us now relate the $\mathcal{R}$-algebra $\mathcal{S}(\mathbb{T}^2)$ to the quantum torus

$$\mathcal{T} = \mathcal{R}\langle M^{\pm 1}, L^{\pm 1}\rangle/(LM - t^2 ML).$$

Let $\varphi : \mathcal{T} \to \mathcal{T}$ be the involution defined by $\varphi(M^k L^l) := M^{-k} L^{-l}$. Frohman and Gelca [**FG**] showed that there is a unique algebra homomorphism $\Upsilon : \mathcal{S}(\mathbb{T}^2) \to \mathcal{T}$ such that

$$\Upsilon(\mu) = -(M + M^{-1}), \quad \Upsilon(\lambda) = -(L + L^{-1}),$$

and Υ maps $\mathcal{S}(\mathbb{T}^2)$ isomorphically onto the symmetric part $\mathcal{T}^\varphi$.

More explicitly,

$$\Upsilon(\lambda_{k,l}) := (-1)^{k+l} t^{kl} (M^k L^l + M^{-k} L^{-l}).$$

The fact that $\mathcal{S}(\mathbb{T}^2)$ and $\mathcal{T}^\varphi$ are isomorphic algebras was also proved by Sallenave [**Sal**].

In the consequence, we will often identify $\mathcal{S}(\mathbb{T}^2)$ with $\mathcal{T}^\varphi$ using the map Υ.

***Example 6.7* (Two-punctured disk and punctured torus).** — Let $F_{0,3}$ be the disk with two points removed, and $F_{1,1}$ be the torus with one point removed.

Then $F_{0,3} \times [0,1] \cong F_{1,1} \times [0,1]$. Hence $\mathcal{S}(F_{0,3})$ and $\mathcal{S}(F_{1,1})$ are isomorphic as $\mathcal{R}$-modules. However, as $\mathcal{R}$-algebras, they are different. In fact, while $\mathcal{S}(F_{0,3})$ is commutative, $\mathcal{S}(F_{1,1})$ is not.

***Exercise 6.8*.** — Using the $\mathcal{R}$-basis of $\mathcal{S}(F_{0,3})$ consisting of embedded links in $F_{0,3}$ without trivial components to show that $\mathcal{S}(F_{0,3})$ is the commutative polynomial algebra $\mathcal{R}[x, x', y]$, where x, x' are small loops in $F_{0,3}$ surrounding the punctured points, and y is a loop in $F_{0,3}$ parallel to the boundary of the disk.

The skein algebra of the punctured torus $F_{1,1}$ is a quantization (non-commutative) algebra of the Lie algebra so_3 and has the following presentation [**BP**] :

$$\mathcal{S}(F_{1,1}) = \mathcal{R}\langle x_0, x_1, x_2\rangle/(t x_i x_{i+1} - t^{-1} x_{i+1} x_i = (t^2 - t^{-2}) x_{i+2}),$$

where the indices are taken modulo 3.

Many results and proofs in the skein theory reduce to calculations involving skein algebras of $F_{0,3}$ and $F_{1,1}$, see e.g. [**BW, Le3**].

6.3. The peripheral and orthogonal ideals. — Suppose K is an oriented knot in S^3. Let $N(K)$ be a tubular neighborhood of K in S^3, and X the closure of $S^3 \setminus N(K)$. Then $\partial(N(K)) = \partial(X) \cong \mathbb{T}^2$. There is a standard choice of an oriented meridian μ and an oriented longitude λ on $\mathbb{T}^2$ such that the linking number between the longitude and K is zero and the linking number between the knot and the meridian is 1. We use this pair (μ, λ) to define the map Υ as in Example 6.6, identifying $\mathcal{S}(\mathbb{T}^2)$ with $\mathcal{T}^\varphi$.

The torus $\mathbb{T}^2 = \partial(N(K))$ is the common boundary of $N(K)$ and X. We can consider $\mathcal{S}(X)$ as a left $\mathcal{S}(\mathbb{T}^2)$-module and $\mathcal{S}(N(K))$ as a right $\mathcal{S}(\mathbb{T}^2)$-module. There is a $\mathcal{R}$-bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{S}(N(K)) \otimes_{\mathcal{S}(\mathbb{T}^2)} \mathcal{S}(X) \to \mathcal{S}(S^3) \equiv \mathcal{R} \tag{39}$$

given by $\langle \ell', \ell'' \rangle := \langle \ell' \cup \ell'' \rangle$, where ℓ' and ℓ'' are links in respectively $N(K)$ and X. Note that if $\ell \in \mathcal{S}(\mathbb{T}^2)$, then

$$\langle \ell' \cdot \ell, \ell'' \rangle = \langle \ell', \ell \cdot \ell'' \rangle.$$

Let us define the $\mathcal{S}(\mathbb{T}^2)$-linear map

$$\Theta : \mathcal{S}(\mathbb{T}^2) \to \mathcal{S}(X), \quad \Theta(\ell) := \ell \cdot \emptyset.$$

In other words, $\Theta : \mathcal{S}(\mathbb{T}^2) \to \mathcal{S}(X)$ is the homomorphism induced from the inclusion $\mathbb{T}^2 \hookrightarrow X$.

The kernel $\mathcal{P} := \ker \Theta$ is called the *quantum peripheral ideal*, first introduced in [**FGL**]. In [**FGL, Ge**], it was proved that every element in $\mathcal{P}$ gives rise to a recurrence relation for the colored Jones polynomial. We will present a refinement of this result here.

The *orthogonal ideal* $\mathcal{O}$ in [**FGL**] is defined as $\Theta^{-1}(\mathcal{S}(N(K))^\perp)$, where $\mathcal{S}(N(K))^\perp$ is the orthogonal complement of $\mathcal{S}(N(K))$ with respect to the bilinear form (39). In other words,

$$\mathcal{O} := \{\ell \in \mathcal{S}(\partial X) \quad | \quad \langle \ell', \Theta(\ell) \rangle = 0 \quad \text{for every } \ell' \in \mathcal{S}(N(K))\}.$$

It is clear that $\mathcal{O}$ is a left ideal of $\mathcal{S}(\partial X) \equiv \mathcal{T}^\varphi$. In [**FGL**], $\mathcal{O}$ was called the formal ideal. Since $\mathcal{P} = \Theta^{-1}(\{0\})$ and $\mathcal{O} = \Theta^{-1}(\mathcal{S}(N(K))^\perp)$, one has

$$\mathcal{P} \subset \mathcal{O}. \tag{40}$$

Conjecture 2. — *For every knot, one has $\mathcal{P} = \mathcal{O}$.*

According to [**Le2**], if the conjecture holds, then the colored Jones polynomial distinguish the unknot from other knots.

Exercise 6.9. — Calculate $\mathcal{P}, \mathcal{O}, \mathcal{A}$ for the unknot.

6.4. Recurrence relations from orthogonal ideal. — As mentioned above, the skein algebra of the torus $\mathcal{S}(\mathbb{T}^2)$ can be identified with $\mathcal{T}^\varphi$ via the $\mathcal{R}$-algebra isomorphism Υ sending μ, λ and $\lambda_{1,1}$ to respectively $-(M + M^{-1}), -(L + L^{-1})$ and $t(ML + M^{-1}L^{-1})$. We will use this identification.

Fix a knot K. We use $\mathcal{A}$ to denote $\mathcal{A}_K$, the recurrence ideal of the knot K. The following proposition says that under the above identification, every element in the orthogonal ideal, a fortiori every element in the quantum peripheral ideal, is a recurrence relation for the colored Jones polynomial.

Proposition 6.10. — *For every knot one has*

$$\mathcal{P} \subset \mathcal{O} \subset \mathcal{A}.$$

Actually,

$$\mathcal{O} = \mathcal{A} \cap \mathcal{T}^\varphi. \tag{41}$$

To prove the above proposition, we first prove the following.

***Proposition 6.11* (see [LT1]).** — *For any skein element $\ell \in \mathcal{S}(\mathbb{T}^2)$ and any $n \in \mathbb{Z}$ one has*

$$(-1)^{n-1} \langle S_{n-1}(K), \Theta(\ell) \rangle = \ell \cdot J_K(n) \tag{42}$$

Here on the left hand side, ℓ is an element of $\mathcal{S}(\partial X)$, and on the right hand side ℓ is an element of $\mathcal{T}^\varphi \subset \mathcal{T}$, which acts on $\mathrm{Map}(\mathbb{Z}, \mathbb{Z}[t^{\pm 1}])$.

Démonstration. — From the recurrence relation for the Chebyshev polynomial, we have

$$\langle S_{n-1}(K)\cdot\lambda,\,\emptyset\rangle = \langle S_n(K)+S_{n-2}(K),\emptyset\rangle \tag{43}$$

The following identities can be proved using the Jones-Wenzl idempotent (see e.g. [**Li, Oh**])

$$\langle S_{n-1}(K)\cdot\mu,\,\emptyset\rangle = (-t^{2n}-t^{-2n})\langle S_{n-1}(K),\emptyset\rangle \tag{44}$$

$$\langle S_{n-1}(K)\cdot\lambda_{1,1},\,\emptyset\rangle = -\langle t^{2n+1}S_n(K)+t^{-2n+1}S_{n-2}(K),\emptyset\rangle. \tag{45}$$

Alternatively, a dedicated reader can prove Identities (44) and (45) in tandem using induction on n. (One has to begin with $n=0$, then move to $n>0$ and $n<0$ using the recurrence relation of the Chebyshev polynomial.)

By definition $J_K(n)=(-1)^{n-1}\langle S_{n-1}(\lambda),\emptyset\rangle$ and

$$(MJ_K)(n)=t^{2n}J_K(n),\quad (LJ_K)(n)=J_K(n+1).$$

Hence, Identities (43)–(45) can be rewritten as

$$\begin{aligned}(-1)^{n-1}\langle S_{n-1}(K),\,\Theta(\lambda)\rangle &= -(L+L^{-1})J_K(n)=\lambda\cdot J_K(n)\\ (-1)^{n-1}\langle S_{n-1}(K),\,\Theta(\mu)\rangle &= -(M+M^{-1})J_K(n)=\mu\cdot J_K(n)\\ (-1)^{n-1}\langle S_{n-1}(K),\,\Theta(\lambda_{1,1})\rangle &= t(ML+M^{-1}L^{-1})J_K(n)=\lambda_{1,1}\cdot J_K(n),\end{aligned}$$

which means the proposition holds true for $\ell=\lambda,\mu$, or $\lambda_{1,1}$. Since $\mathcal{S}(\mathbb{T}^2)$ is generated by μ,λ and $\lambda_{1,1}$, we conclude that the proposition holds for all $\ell\in\mathcal{S}(\mathbb{T}^2)$. □

Proof of Proposition 6.10. — We already have $\mathcal{P}\subset\mathcal{O}$, see (40). We now show $\mathcal{O}\subset\mathcal{A}$. Suppose $\ell\in\mathcal{O}$. Then the left hand side of (42) is 0. The right hand side of (42) is 0 means that $\ell\in\mathcal{A}$. Thus, $\mathcal{O}\subset\mathcal{A}$.

Next we show that $\mathcal{O}=\mathcal{A}_K\cap\mathcal{T}^\varphi$.

Since $\{S_n(K)\mid n\in\mathbb{Z}\}$ spans the skein module $\mathcal{S}(N(K))$, Proposition 6.11 implies that

$$\begin{aligned}\mathcal{O} &= \{\ell\in\mathcal{S}(\partial X)\mid \langle\ell',\Theta(\ell)\rangle=0\quad\text{for all } \ell'\in\mathcal{S}(N(K))\}\\ &= \{\ell\in\mathcal{S}(\partial X)\mid \langle S_n(K),\Theta(\ell)\rangle=0\quad\text{for all integers } n\}\\ &= \{\ell\in\mathcal{S}(\partial X)=\mathcal{T}^\varphi\mid \ell\cdot J_K(n)=0\quad\text{for all integers } n\}\\ &= \{\ell\in\mathcal{S}(\partial X)=\mathcal{T}^\varphi\mid \ell\in\mathcal{A}\}\\ &= \mathcal{A}\cap\mathcal{T}^\varphi.\end{aligned}$$

□

Remark 6.12. — Equation (41) was obtained in [**Ga1**] by another method. We present here a more geometric proof, using properties of the action of the longitude and the meridian on the Chebyshev polynomials.

By Proposition 6.10, every element of the quantum peripheral ideal $\mathcal{P}=\ker\Theta$ gives a recurrence relation for the colored Jones polynomial. Hence, one could prove Theorem 5.4 about the existence of recurrence relations for the colored Jones polynomials if one can show that $\mathcal{P}\neq 0$ for any knot.

Conjecture 3. — *For any knot K, the quantum peripheral ideal $\mathcal{P}\neq 0$.*

The conjecture would give a skein theoretic proof of Theorem 5.4. We have the following simpler result.

Proposition 6.13. — *Suppose for a knot K the skein module $\mathcal{S}(X)$ has finite rank over the ring $\mathcal{R}[M+M^{-1}]$. Then $\mathcal{P}\neq 0$.*

Démonstration. — As a module over $\mathcal{R}[M+M^{-1}]=\mathcal{R}[M^{\pm1}]^\varphi$, a Noetherian commutative ring, the quantum torus $\mathcal{T}$ has infinite rank, i.e. it cannot be spanned by a finite number of elements. Hence, if $\mathcal{S}(X)$ has finite rank over $\mathcal{R}[M+M^{-1}]$, then $\mathcal{P}=\ker(\Theta:\mathcal{T}\to\mathcal{S}(X))$ cannot be 0. In this case K has a non-trivial recursion relation coming from $\mathcal{P}$. □

6.5. The character variety of a group. — Now we change the course and discuss relations between skein modules and character varieties. Let us first recall the $SL_2(\mathbb{C})$-character variety of a group. For details, see [**CS, BH, LM**].

For an algebraic set Y we will use the notation $\mathbb{C}[Y]$ to denote the ring of regular functions on Y. Then $\mathbb{C}[Y]$ is a $\mathbb{C}$-algebra, which is finitely-generated and reduced, i.e. its nil-radical is 0. Conversely, every finitely-generated reduced $\mathbb{C}$-algebra is the ring of regular functions of certain algebraic set. For details, see e.g. [**Sh**].

Suppose G is a finitely presented group. The set $\mathrm{Hom}(G, SL_2(\mathbb{C}))$ of all representations of G into $SL_2(\mathbb{C})$ is an algebraic set defined over $\mathbb{C}$, on which $SL_2(\mathbb{C})$ acts by conjugation. Here is a description of $\mathrm{Hom}(G, SL_2(\mathbb{C}))$ as an algebraic set. Suppose G has a presentation

$$G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle,$$

where each r_l is a monomial in positive powers of generators a_j. Such a presentation always exits. Consider $4n$ variables $(A_k)_{ij}$ with $k = 1, \dots, n$, and $i, j = 1, 2$. Let $I(G)$ be the ideals in the rings of $\mathbb{C}$-polynomials in the $4n$ variables $(A_k)_{ij}$ generated by the following $4m + n$ polynomials

$$\begin{aligned} &\det A_k - 1, \quad k = 1, \dots, n \\ &(R_l)_{ij} - \delta_{ij}, \quad l = 1, \dots, m, \ i, j = 1, 2, \end{aligned}$$

where A_k is the 2×2 matrix with entries $(A_k)_{ij}$ and R_l is the 2×2 matrix $R_l = r_l(A_1, \dots, A_n)$.

Then $\mathrm{Hom}(G, SL_2(\mathbb{C}))$ is the zero set of $I(G)$. The ideal of all polynomials in $\mathbb{C}[(A_k)_{ij}]$ vanishing on $\mathrm{Hom}(G, SL_2(\mathbb{C}))$ is the radical $\sqrt{I(G)}$ of $I(G)$. Hence, we have

$$\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))] = \mathrm{Rep}(G)/\sqrt{I(G)}.$$

The ring

$$\mathrm{Rep}(G) = \mathbb{C}[(A_k)_{ij}]/I(G)$$

is known as the *universal representation ring* of G, see below. It is known that $\mathrm{Rep}(G)$ does not depend on particular presentations of G.

The universal representation ring $\mathrm{Rep}(G) = \mathbb{C}[(A_k)_{ij}]/I(G)$ is reduced if and only the ideal $I(G)$ is radical, i.e. $\sqrt{I(G)} = I(G)$. If $I(G)$ is radical, or the same as $\mathrm{Rep}(G)$ is reduced, then the ring of regular functions $\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))]$ of $\mathrm{Hom}(G, SL_2(\mathbb{C}))$ is $\mathrm{Rep}(G)$.

The set-theoretic quotient of $\mathrm{Hom}(G, SL_2(\mathbb{C}))$ by the conjugate action of $SL_2(\mathbb{C})$ does not have good topological properties, because in general the orbits are not closed (in the Zariski topology or in the $\mathbb{C}$-topology). A better quotient $\chi(G)$, called the *algebro-geometric quotient* with a surjection

$$p : \mathrm{Hom}(G, SL_2(\mathbb{C})) \twoheadrightarrow \chi(G),$$

has the structure of an algebraic set, and p is a regular map. Two representations have the same image under p if and only if they have the same character. This means p descends to a bijection between the set of all $SL_2(\mathbb{C})$-characters of G and $\chi(G)$. For this reason $\chi(G)$ is usually called the *character variety* of G. The reader should be careful with this terminology, as $\chi(G)$ might have many components and is not an affine variety as defined in most textbooks in algebraic geometry.

Formally one can define $\chi(G)$ as follows. As $SL_2(\mathbb{C})$ acts on $\mathrm{Hom}(G, SL_2(\mathbb{C}))$, it acts on $\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))]$. The subring $\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))]^{SL_2(\mathbb{C})}$ of $SL_2(\mathbb{C})$-invariant elements is finitely-generated, according to Hilbert's theorem on finiteness of invariants. Besides, $\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))]^{SL_2(\mathbb{C})}$, as a subring of $\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))]$, is reduced. Hence, $\mathbb{C}[\mathrm{Hom}(G, SL_2(\mathbb{C}))]^{SL_2(\mathbb{C})}$ is the ring of regular functions of an algebraic set, which is $\chi(G)$.

For a manifold Y we use $\chi(Y)$ also to denote $\chi(\pi_1(Y))$.

The character variety has played an important role in geometric topology.

Exercise 6.14. — Let $\mathbb{Z} = \langle a \rangle$. Show that the map $\mathrm{Hom}(\mathbb{Z}, SL_2(\mathbb{C})) \to \mathbb{C}$, $\rho \to \mathrm{tr}(\rho(a))$ descends to an isomorphism between $\chi(\mathbb{Z})$ and $\mathbb{C}$.

The next example is important for us.

Example 6.15. — Suppose $G = \mathbb{Z}^2$, the free abelian group with two generators μ, λ. There is an isomorphism between $\chi(G)$ and $(\mathbb{C}^*)^2/\tau$, where $(\mathbb{C}^*)^2$ is the set of non-zero complex number pairs (L, M) and τ is the involution $\tau(M, L) := (M^{-1}, L^{-1})$, as follows : Suppose x is the character of $\rho \in \mathrm{Hom}(\mathbb{Z}^2, SL_2(\mathbb{C}))$. Then ρ is conjugate to a representation ρ' where $\rho'(\gamma)$ is upper diagonal for every $\gamma \in \mathbb{Z}$. Let M and L be the upper left entry of the matrices $\rho'(\mu)$ and $\rho(\lambda)$ respectively. Then the map $x \to (M, L)$, from $\chi(\mathbb{Z}^2)$ to $(\mathbb{C}^*)^2/\tau$, is well-defined and is an isomorphism of algebraic sets.

Exercise 6.16. — Show that $\mathbb{C}[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\varphi$, the φ-invariant subspace of $\mathfrak{t} := \mathbb{C}[L^{\pm1}, M^{\pm1}]$, where $\varphi(M^k L^l) = M^{-k} L^{-l}$.

Exercise 6.17. — Find the character variety of $\mathbb{Z}^n$.

For details of the next two examples, see [**LM**].

Example 6.18. — Suppose $G = \langle a, b \rangle$, the free group on 2 generators a, b. The map $\chi(G) \to \mathbb{C}^3$, given by $x \to (\mathrm{tr}(\rho(a)), \mathrm{tr}(\rho(b)), \mathrm{tr}(\rho(ab)))$, where ρ is a representation with character x, is an isomorphism.

Example 6.19. — Let $F_n = \langle a_1, \dots, a_n \rangle$, the free group on n generators. It is clear that $\mathrm{Hom}(F_n, SL_2(\mathbb{C})) \cong (SL_2(\mathbb{C}))^n$, which has dimension $3n$ (over $\mathbb{C}$). Since the dimension of $SL_2(\mathbb{C})$ is 3, one can show that the dimension of $\chi(F_n)$ is $3n - 3$.

As an affine algebraic set, there is an embedding $\chi(F_n) \hookrightarrow \mathbb{C}^k$ for some k. In fact, there are k elements $g_1, \dots g_k$ of F_n such that their traces generate the ring of characters of F_n, i.e. for every element $g \in F_n$ there exists a polynomial $\mathbf{P}_g \in \mathbb{C}[x_1, \dots, x_k]$ such that for $y \in \chi(F_n)$, one has $y(g) = \mathbf{P}_g(y(g_1), \cdots, y(g_k))$. The map

$$y \to (y(g_1), \cdots, y(g_k)) \in \mathbb{C}^k$$

is an embedding of $\chi(F_n)$ into $\mathbb{C}^k$.

6.6. Functorial properties. — Suppose $f : G \to H$ is a group homomorphism. Then f induced a regular map $\mathrm{Hom}(H, SL_2(\mathbb{C})) \to \mathrm{Hom}(G, SL_2(\mathbb{C}))$ which descends to a regular map

$$f^* : \chi(H) \to \chi(G).$$

One can easily show that the assignment $G \to \chi(G)$ is a contravariant functor, i.e. $\mathrm{id}^* = \mathrm{id}$ and$(fg)^* = g^* f^*$.

Taking the dual, we get a covariant function $G \to \mathbb{C}[\chi(G)]$, with $f_* : \mathbb{C}[\chi(G)] \to \mathbb{C}[\chi(H)]$.

6.7. The universal character ring. — Suppose G is a group, A is $\mathbb{C}$-algebra, and $\rho : G \to SL_2(A)$ is a representation. Any $\mathbb{C}$-algebra homomorphism $f : A \to A'$ induces an algebra homomorphism $f_\# : SL_2(A) \to SL_2(A')$ and

$$f_\# \circ \rho : G \to SL_2(A')$$

is called the representation derived from ρ via f.

For every finitely presented group G there is a commutative $\mathbb{C}$-algebra $\mathrm{Rep}(G)$, called the *universal representation algebra*, and the *universal representation*

$$\Psi : G \to SL_2(\mathrm{Rep}(G))$$

such that for every $\mathbb{C}$-algebra A and every representation $\rho : G \to SL_2(A)$, there is a unique $\mathbb{C}$-algebra homomorphism $f : \mathrm{Rep}(G) \to A$ such that ρ is derived from Ψ via f. For details, see [**LM, Si**].

The ring $\mathrm{Rep}(G)$, described in the previous section, is actually the universal representation of G. The universal representation is the obvious one, $\Psi(a_i) = A_i$.

Exercise 6.20. — Prove the above statements.

The group SL_2 acts by conjugation on $\mathrm{Rep}(G)$, and the subring $\mathrm{Rep}(G)^{SL_2(\mathbb{C})}$ of fixed points is called the *universal character ring* of G.

The universal character can also be described in a more explicit way as follows. Suppose G has a presentation

$$G = \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle.$$

By Example 6.19, there are a finite number of elements $g_1, \cdots, g_k$ in the free group $F_n = \langle a_1, \dots, a_n \rangle$ such that for every element $g \in F_n$ there exists a polynomial $\mathbf{P}_g \in \mathbb{C}[x_1, \dots, x_k]$ with the property $y(g) = \mathbf{P}_g(y(g_1), \cdots, y(g_k))$ for any character $y \in \chi(F_n)$.

The universal character ring of G is isomorphic the quotient of the ring $\mathbb{C}[x_1, \cdots, x_k]$ by the ideal generated by all expressions of the form $\mathbf{P}_g - \mathbf{P}_{g'}$, where g and g' are any two elements of F_n which have the same image in G.

6.8. Skein modules at ± 1 and character variety. — For a non-zero complex number ξ let $\mathcal{S}_\xi(Y)$ be the skein module of Y at $t = \xi$, i.e.

$$\mathcal{S}_\xi(Y) = \mathcal{S}(Y)/(t - \xi) = \mathcal{S}(Y) \otimes_{\mathcal{R}} \mathbb{C},$$

where $\mathbb{C}$ is considered as a $\mathcal{R}$-module by setting $t \to \xi$. Then $\mathcal{S}_\xi(Y)$ is a vector space over $\mathbb{C}$.

When ξ is a root of unity, $\mathcal{S}_\xi(Y)$ plays important role in quantum topology. For example, the $SU(2)$ topological quantum field theory theory can be constructed using $\mathcal{S}_\xi(Y)$.

The cases $\xi = \pm 1$ are more special, because $\mathcal{S}_{\pm 1}(Y)$ have a natural commutative algebra structure where the product of two links in $\mathcal{S}_{\pm}(Y)$ is their disjoint union.

Exercise 6.21. — Show that when $\xi = \pm 1$, this product is well-defined.

Suppose Y is a compact oriented 3-manifold. Using a triangulation of Y one can show that $\mathcal{S}_{-1}(Y)$ is a finitely-generated $\mathbb{C}$-algebra. Let $\sqrt{0}$ be the nil-radical of $\mathcal{S}_{-1}(Y)$. Then $\mathcal{S}_{-1}(Y)/\sqrt{0}$ is a reduced finitely generated $\mathbb{C}$-algebra. Hence, $\mathcal{S}_{-1}(Y)/\sqrt{0}$ is isomorphic to the ring of regular functions of a certain algebraic set.

An important result [**Bul, PS**] in the theory of skein modules is that $\mathcal{S}_{-1}(Y)$ is naturally isomorphic to the universal character ring of $\pi_1(Y)$, and $\mathcal{S}_{-1}(Y)/\sqrt{0}$ is naturally isomorphic to the character ring $\mathbb{C}[\chi(Y)]$, or the ring of regular functions of the character variety of $\pi_1(Y)$.

The isomorphism between $f : \mathcal{S}_{-1}(Y)/\sqrt{0} \to \mathbb{C}[\chi(Y)]$ is defines as follows. As the set of framed knots in Y generate $\mathcal{S}_{-1}(Y)$ as an algebra, it is enough to define $f(K)$ for every framed knot $K \in Y$. Suppose $x \in \chi(Y)$, considered as a class function on $\pi_1(Y)$. Then $f(K) \in \mathbb{C}[\chi(Y)]$ is the regular function on $\chi(Y)$ given by

$$f(K)(x) = -x(K),$$

where on the right hand side we consider K as an element in $\pi_1(Y)$, which is well-defined up to conjugation.

In many cases $\mathcal{S}_{-1}(Y)$ is reduced, i.e. its nilradical is zero, and hence $\mathcal{S}_{-1}(Y)$ is exactly the ring of regular functions on the SL_2-character variety of $\pi_1(Y)$. For example, this is the case when Y is the complement of the torus knots (see [**Mar**]), when Y is the complement of a two-bridge knot/link (see [**Le2, PS, LT1**]), or when Y is the complement of the $(-2, 3, 2n+1)$-pretzel knot for any integer n (see [**LT2**]). We have the following conjecture.

Conjecture 4. — *For every knot K in S^3, the universal SL_2-character ring of $S^3 \setminus K$ is reduced. In other words, the skein algebra $\mathcal{S}_{-1}(S^3 \setminus K)$ is reduced.*

The algebra $\mathcal{S}_1(Y)$ is isomorphic to $\mathcal{S}_{-1}(Y)$, and hence is isomorphic to the universal character ring. Barrett [**Bar**] shows that every spin structure of Y defines an algebra isomorphism from $\mathcal{S}_1(Y)$ to $\mathcal{S}_{-1}(Y)$. However, it seems there is no natural algebra isomorphism between $\mathcal{S}_1(Y)$ and $\mathcal{S}_{-1}(Y)$, as there is no natural spin structure on 3-manifolds.

6.9. Skein modules at roots of 1. — Suppose $\xi \in \mathbb{C}$ is a root of unity of order $2N$, where N is an odd number.

There is a unique action of $\mathcal{S}_{-1}(Y)$ on $\mathcal{S}_\xi(Y)$ such that if ℓ, ℓ' are disjoint framed links in Y, then

$$\ell \cdot \ell' = T_N(\ell) \cup \ell',$$

where T_N is the N-th Chebyshev polynomial of type 1, defined in Subsection 4.2. On the left hand side ℓ is considered as an element of $\mathcal{S}_{-1}(Y)$, ℓ' is considered as an element of $\mathcal{S}_\xi(Y)$. On the right hand side both ℓ, ℓ' are considered as elements of $\mathcal{S}_\xi(Y)$. In [**Le3**], it was proved that this gives rise to an action of $\mathcal{S}_{-1}(Y)$ on $\mathcal{S}_\xi(Y)$, which is an extension of results of [**BW**] for skein algebras of surfaces.

7. AJ conjecture

Here we discuss the AJ conjecture and sketch a proof of it for a class of two-bridge knots and pretzel knots. Most of the results here are taken from [**Le2, LT2**].

7.1. The A-polynomial. — Suppose K is a knot in S^3, and $N(K)$ is a tubular neighborhood of K. Let X be the closure of $S^3 \setminus N(K)$. The boundary of X is a torus whose fundamental group is free abelian of rank two. An orientation of K will define a unique pair of an oriented meridian and an oriented longitude such that the linking number between the longitude and the knot is zero, as in Subsection 6.3. The pair provides an identification of $\chi(\partial X)$ and $(\mathbb{C}^*)^2/\tau$, see Example 6.15.

Exercise 7.1. — Show that the above identification does not depend on the orientation of the knot K.

The inclusion $\partial X \hookrightarrow X$ induces the restriction map

$$\vartheta : \chi(X) \longmapsto \chi(\partial X) \equiv (\mathbb{C}^*)^2/\tau \tag{46}$$

Let Z be the image of ρ and $\hat{Z} \subset (\mathbb{C}^*)^2$ the lift of Z under the projection $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2/\tau$. The Zariski closure of $\hat{Z} \subset (\mathbb{C}^*)^2 \subset \mathbb{C}^2$ in $\mathbb{C}^2$ is an algebraic set consisting of components of dimension 0 or 1. The union of all the one-dimension components is defined by a reduced polynomial $A_K \in \mathbb{Z}[M, L]$, whose coefficients are co-prime. Here A_K is reduced means A_K does not have repeated factor. Note that A_K is defined up to ± 1. We call A_K the *A-polynomial* of K. It is known that A_K is always divisible by $L - 1$. The A-polynomial in this paper is actually equal to $L - 1$ times the A-polynomial defined in [**CCGLS**].

The A-polynomial is an important geometric invariant. The slops of the Newton polygon of A_K are boundary slopes of the knot. The A-polynomial distinguishes the unknot from other knots, see [**DG, BZ**].

For a hyperbolic knot [**Th**], the character of a discrete faithful SL_2-representation is always a smooth point of the character variety, see e.g. [**Po**]. A component of the character variety containing the character of a discrete faithful representation is called a *geometric component*. By a result of Thurston, the complex dimension of each geometric component is 1. For knots in S^3 there are at most 4 geometric components, see e.g. [**Du**]. There is no known example of knots with more than one geometric components.

An important result of Dunfield [**Du**] that we will use is that the map ϑ in (46), when restricted to a geometric component, is a birational equivalence onto its image.

7.2. AJ conjecture. —

Definition 3. — *Suppose $f, g \in \mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$. Then*

(a) f is M-essentially equal to g if there are non-zero $a, b \in \mathbb{C}[M^{\pm 1}]$ such that $af = bg$.

(b) f is M-essentially divisible by g if there is a non-zero $a \in \mathbb{C}[M^{\pm 1}]$ such that af is divisible by g.

Suppose K is an unframed knot in S^3. Let α_K be the recurrence polynomial of the colored Jones function $J_{K^{(0)}} : \mathbb{Z} \to \mathbb{Z}[t^{\pm 1}]$, where $K^{(0)}$ is the knot K with framing 0. By Proposition 5.6, α_K has only even powers in t and even powers in M.

Garoufalidis [**Ga2**] formulated the following conjecture (see also [**FGL, Ge**]).

Conjecture 5. — **(AJ conjecture)** *For every knot K, $\alpha_K|_{t=\pm 1}$ is M-essentially equal to the A-polynomial.*

The AJ conjecture gives a very deep relation between the colored Jones polynomial and the fundamental group. Some authors also call the recurrence polynomial α_K the quantum A-polynomial.

Example 7.2. — For the right-handed trefoil, α_K is given by (36). One has

$$\alpha_K \mid_{t=-1} = (M^4 - 1)(L - 1)(LM^6 + 1) = (M^4 - 1)\, A_K(L, M),$$

and the conjecture holds for the trefoil.

The A-polynomial is difficult to calculate, the recurrence polynomial is even more difficult to calculate. There are only a few simple knots for which the AJ conjecture can be verified by direct calculation. For torus knots, the AJ conjecture was verified in [**Hi, Tr**] using explicit formulas of the colored Jones polynomials. What we propose here is a more conceptual proof of the AJ conjecture for another class of knots for which no explicit formulas of the colored Jones polynomial are known.

7.3. Results. — Suppose K is a knot in $\mathbb{R}^3 \subset S^3$. As usual X is the closure of $S^3 \setminus N(K)$, where $N(K)$ is a tubular neighborhood of K. Then $\mathcal{S}(X)$ is a left $\mathcal{S}(\partial X)$-module. We already know that $\mathcal{S}(\partial X) = \mathcal{S}(\mathbb{T}^2) = \mathcal{T}^\varphi$, where $\mathcal{T} = \mathcal{R}\langle L^{\pm 1}, M^{\pm 1}\rangle/(LM = t^2 ML)$ is the quantum torus and φ is the algebra involution of $\mathcal{T}$ given by $\varphi(M^a L^b) = M^{-a}L^{-b}$.

Let $\mathcal{M} = \mathcal{R}[M^{\pm 1}] \subset \mathcal{T}$. Then $\mathcal{M}^\varphi = \mathcal{R}[M + M^{-1}] \subset \mathcal{T}^\varphi$. Since $\mathcal{S}(X)$ is a $\mathcal{T}^\varphi$-module, it is a module over $\mathcal{M}^\varphi$.

Theorem 7.3 **(See [LT2]).** — *Suppose K is a knot satisfying all the following conditions :*

(i) K is hyperbolic,

(ii) the SL_2-character variety of $\pi_1(S^3 \setminus K)$ consists of two irreducible components (one abelian and one non-abelian),

(iii) the universal SL_2-character ring of $\pi_1(S^3 \setminus K)$ is reduced,

(iv) the skein module $\mathcal{S}(X)$ is finitely generated over $\mathcal{M}^\varphi$, and

(v) the recurrence polynomial of K has L-degree greater than 1.

Then the AJ conjecture holds true for K.

Note that if K is adequate, then (v) holds. If K is non-torus alternating, then (i) and (v) hold. On the other hand, if K is torus, then it is known that the AJ conjecture holds [**Hi, Tr**].

Actually, the conclusion of the theorem still holds true if conditions (iii) and (iv) are replaced by weaker conditions, see below.

Theorem 7.4 **(See [LT2]).** — *The following knots satisfy all the conditions (i)–(v) of Theorem 7.3 and hence the AJ conjecture holds true for them.*

(a) All pretzel knots of type $(-2, 3, 6n \pm 1)$, $n \in \mathbb{Z}$.

(b) All two-bridge knots for which the SL_2-character variety has exactly two irreducible components ; these include

- *all double twist knots of the form $\mathcal{J}(k, l)$ (see Figure 11) with $k \neq l$*
- *all two-bridge knots $\mathfrak{b}(p, m)$ with $m = 3$, and*
- *all two-bridge knots $\mathfrak{b}(p, m)$ with p prime and $\gcd(\frac{p-1}{2}, \frac{m-1}{2}) = 1$.*

Here we use the notation $\mathfrak{b}(p, m)$ for two bridge knots from [**BZ**]. The fact that the character varieties of pretzel knots $(-2, 3, 6n \pm 1)$ and double twist knots have exactly 2 components was proved in [**MPL**] and in [**Mat**].

Actually, (b) can be strengthen as follows : if the non-abelian character variety of a two-bridge knot K is irreducible over $\mathbb{Z}$, then the AJ conjecture holds for K (joint work with X. Zhang).

Remark 7.5. — Besides the infinitely many cases of two-bridge knots listed in Theorem 7.4, explicit calculations seem to confirm that "most two-bridge knots" satisfy the conditions of Theorem 7.3 and hence AJ conjecture holds for them. In fact, among 155 $\mathfrak{b}(p, m)$ with $p < 45$, only 9 hyperbolic knots $\mathfrak{b}(15, 11)$, $\mathfrak{b}(21, 13)$, $\mathfrak{b}(27, 5)$, $\mathfrak{b}(27, 17)$, $\mathfrak{b}(27, 19)$, $\mathfrak{b}(33, 5)$, $\mathfrak{b}(33, 13)$, $\mathfrak{b}(33, 23)$, and $\mathfrak{b}(35, 29)$ do not satisfy the condition (ii) of Theorem 7.3. Thus, the AJ conjecture holds for all two-bridge knots $\mathfrak{b}(p, m)$ with $p < 45$ except for these 9 knots. Using explicit formula, Garoufalidis and Koutchan [**GK**] showed that the AJ conjecture holds for $\mathfrak{b}(15, 11)$.

In the rest of the section we explain main ideas of the proof of Theorem 7.3.

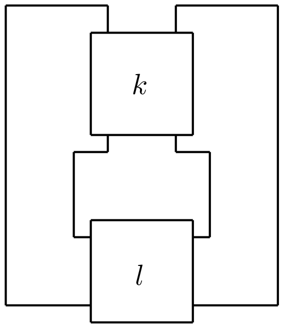


FIGURE 11. The double twist knot $\mathcal{J}(k,l)$. Here k and l denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists .

7.4. A sibling of the A-polynomial. — It is instructive to see the dual picture in the construction of the A-polynomial.

Recall that $\mathbb{C}[(\mathbb{C}^*)^2/\tau] = \mathfrak{t}^\varphi$, the φ-invariant subspace of $\mathfrak{t} = \mathbb{C}[L^{\pm 1}, M^{\pm 1}]$, where $\varphi(M^k L^l) = M^{-k}L^{-l}$.

The map ϑ in Subsection 7.1 has a dual, which is an algebra homomorphism

$$\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\varphi \longrightarrow \mathbb{C}[\chi(X)].$$

We will call the kernel $\mathfrak{p}$ of θ the *classical peripheral ideal*, which is is an ideal of $\mathfrak{t}^\varphi$. We have the exact sequence

$$0 \to \mathfrak{p} \to \mathfrak{t}^\varphi \xrightarrow{\theta} \mathbb{C}[\chi(X)]. \tag{47}$$

The ring $\mathfrak{t} = \mathbb{C}[M^{\pm 1}, L^{\pm 1}]$ embeds naturally into the principal ideal domain $\widetilde{\mathfrak{t}} := \mathbb{C}(M)[L^{\pm 1}]$, where $\mathbb{C}(M)$ is the fractional field of $\mathbb{C}[M]$. We have $\mathfrak{t}^\varphi \subset \mathfrak{t} \subset \widetilde{\mathfrak{t}}$. The ideal extension $\widetilde{\mathfrak{p}} := \widetilde{\mathfrak{t}}\mathfrak{p}$ of $\mathfrak{p}$ in $\widetilde{\mathfrak{t}}$ is thus generated by a single polynomial $B_K \in \mathbb{Z}[M, L]$ which has co-prime coefficients and is defined up to a factor $\pm M^k$ with $k \in \mathbb{Z}$. Again B_K can be chosen to have integer coefficients because everything can be defined over $\mathbb{Z}$. We will call B_K the *B-polynomial* of K.

Exercise 7.6. — Show that that the polynomial B_K is M-essentially divisible by A_K. Moreover, their zero sets $\{B_K = 0\}$ and $\{A_K = 0\}$ are equal, up to some lines parallel to the L-axis in the L, M-plane.

7.5. Relation between the A-polynomial and B-polynomial. —

Lemma 7.7. — *The field $\mathbb{C}(M)$ is a flat $\mathbb{C}[M^{\pm 1}]^\varphi$-algebra, and $\widetilde{\mathfrak{t}} = \mathfrak{t}^\varphi \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M)$.*

Démonstration. — The extension from $\mathbb{C}[M^{\pm 1}]^\varphi$ to $\mathbb{C}(M)$ can be done in two steps : The first one is from $\mathbb{C}[M^{\pm 1}]^\varphi$ to $\mathbb{C}[M^{\pm 1}]$ (note that $\mathbb{C}[M^{\pm 1}]$ is free over $\mathbb{C}[M^{\pm 1}]^\varphi$ since $\mathbb{C}[M^{\pm 1}] = \mathbb{C}[M^{\pm 1}]^\varphi \oplus M\mathbb{C}[M^{\pm 1}]^\varphi$) ; the second step is from $\mathbb{C}[M^{\pm 1}]$ to its field of fractions $\mathbb{C}(M)$. Each step is a flat extension, hence $\mathbb{C}(M)$ is flat over $\mathbb{C}[M^{\pm 1}]^\varphi$.

It follows that the extension $(\mathfrak{t}^\varphi \hookrightarrow \mathfrak{t}) \otimes \mathbb{C}(M)$ is still an injection, i.e.

$$\psi : \mathfrak{t}^\varphi \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M) \to \mathfrak{t} \otimes_{\mathbb{C}[M^{\pm 1}]} \mathbb{C}(M) = \widetilde{\mathfrak{t}}, \quad \psi(x \otimes y) = xy,$$

is injective. Let us show that ψ is surjective. For every $n \in \mathbb{Z}$,

$$L^n \quad = \quad \psi\left((ML^n + M^{-1}L^{-n}) \otimes \frac{1}{M - M^{-1}} - (L^n + L^{-n}) \otimes \frac{M^{-1}}{M - M^{-1}}\right).$$

Since $\{L^n \mid n \in \mathbb{Z}\}$ generates $\widetilde{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}]$, ψ is surjective. Thus ψ is an isomorphism. □

Consider the exact sequence (47). The ring $\mathbb{C}[\chi(X)]$ has a $\mathfrak{t}^\varphi$-module structure via the algebra homomorphism $\theta : \mathbb{C}[\chi(\partial X)] \equiv \mathfrak{t}^\varphi \to \mathbb{C}[\chi(X)]$, hence a $\mathbb{C}[M^{\pm 1}]^\varphi$-module structure since $\mathbb{C}[M^{\pm 1}]^\varphi$ is a subring of $\mathfrak{t}^\varphi$. By Lemma 7.7, $\widetilde{\mathfrak{t}} = \mathfrak{t}^\varphi \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M)$. It follows that $\widetilde{\mathfrak{p}} = \mathfrak{p} \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M)$. Hence by taking the tensor product over $\mathbb{C}[M^{\pm 1}]^\varphi$ of the exact sequence (47) with $\mathbb{C}(M)$, we get the exact sequence

$$0 \to \widetilde{\mathfrak{p}} \to \widetilde{\mathfrak{t}} \xrightarrow{\widetilde{\theta}} \widetilde{\mathbb{C}[\chi(X)]}, \tag{48}$$

where $\widetilde{\mathbb{C}[\chi(X)]} := \mathbb{C}[\chi(X)] \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M)$.

Proposition 7.8. — *The B-polynomial B_K does not have repeated factors.*

Exercise 7.9. — Using the fact that $\mathbb{C}[\chi(X)]$ is a reduced $\mathbb{C}$-algebra, show that $\widetilde{\mathbb{C}[\chi(X)]}$ is also reduced. From here prove the proposition.

Combining the proposition and the result of Exercise 7.6, we get the following, which describe the relation between the A-polynomial and the B-polynomial of a knot.

Corollary 7.10. — *For every knot K, the polynomials A_K and B_K are M-essentially the same.*

Since B_K does not have any non-trivial factor in $\mathbb{Z}[M^{\pm 1}]$, we have

$$B_K = \frac{A_K}{M\text{-factor of } A_K}.$$

Here the M-factor of A_K is the maximal factor of A_K belonging to $\mathbb{Z}[M^{\pm 1}]$.

7.6. Idea of proof of Theorem 7.3. — We will write $\mathcal{S}$ for $\mathcal{S}(X)$. Then $\mathfrak{s} = \mathcal{S}_{-1}(X) = \mathcal{S}/(1+t)$ is the universal character ring of $\pi_1(X)$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{T}^\varphi & \xrightarrow{\Theta} & \mathcal{S} \\ {\scriptstyle\varepsilon_1}\big\downarrow & & {\scriptstyle\varepsilon_2}\big\downarrow \\ \mathfrak{t}^\varphi & \xrightarrow{\theta} & \mathfrak{s} \end{array} \tag{49}$$

Here, the vertical maps are the the natural projections $V \to V/(t+1)$, for $V = \mathcal{T}^\varphi$ and $V = \mathcal{S}$.

Let $\overline{\mathcal{M}}$ be the localization of $\mathcal{M} = \mathbb{C}[t^{\pm 1}, M^{\pm 1}]$ at $(1+t)$, i.e.

$$\overline{\mathcal{M}} = \left\{ \frac{f}{g} \;\mid\; f, g \in \mathbb{C}[t^{\pm 1}, M^{\pm 1}], g|_{t=-1} \neq 0 \right\}.$$

Then $\overline{\mathcal{M}}$ is a local ring, and every ideal of $\overline{\mathcal{M}}$ is one of $((1+t)^k), k \in \mathbb{N}$. It is not difficult to show that $\overline{\mathcal{M}}$ is flat over $\mathcal{M}^\varphi$. Similarly, $\mathbb{C}(M)$ is flat over $\mathbb{C}[M^{\pm 1}]^\varphi$, see Lemma 7.7.

Let $\overline{\mathcal{T}} = \mathcal{T}^\varphi \otimes_{\mathcal{M}^\varphi} \overline{\mathcal{M}}$ and $\overline{\mathcal{S}} = \mathcal{S} \otimes_{\mathcal{M}^\varphi} \overline{\mathcal{M}}$. Similarly, let $\bar{\mathfrak{t}} = \mathfrak{t}^\varphi \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M)$ and $\bar{\mathfrak{s}} = \mathfrak{s} \otimes_{\mathbb{C}[M^{\pm 1}]^\varphi} \mathbb{C}(M, L)$. Then one can show that

$$\overline{\mathcal{T}} = \overline{\mathcal{M}}[L^{\pm 1}]$$
$$\bar{\mathfrak{t}} = \mathbb{C}(M)[L^{\pm 1}] = \widetilde{\mathfrak{t}}.$$

From the Diagram (49), one has

$$\begin{array}{ccc} \overline{\mathcal{T}} & \xrightarrow{\overline{\Theta}} & \overline{\mathcal{S}} \\ {\scriptstyle\varepsilon}\big\downarrow & & {\scriptstyle\varepsilon}\big\downarrow \\ \bar{\mathfrak{t}} & \xrightarrow{\bar{\theta}} & \bar{\mathfrak{s}} \end{array} \tag{50}$$

Assume that
(iii') the ring $\bar{\mathfrak{s}}$ is reduced.

This condition is weaker than condition (iii) of Theorem 7.3. Then

$$\bar{\mathfrak{s}} = \mathbb{C}[\chi(X)] \otimes_{\mathfrak{t}^\varphi} \bar{\mathfrak{t}}. \tag{51}$$

According to condition (ii), the character variety of X has two components, one abelian and one non-abelian. Since K is hyperbolic, the non-abelian is the geometric component and it is the only irreducible component of the character variety containing the character of the discrete faithful SL_2 representation. By a result of Dunfield [**Du**], the map from the geometric component onto the character variety of the boundary torus is a birational map on its image. From here and the condition (51) one can show that $\overline{\theta}$ is surjective.

Now assume that

(iv') $\overline{\mathcal{S}}$ is finitely generated over $\overline{\mathcal{M}}$.

This condition is weaker than (iv). Then Nakayama's lemma and Diagram (50) show that $\overline{\Theta}$ is surjective.

We have now the following commutative diagram with exact rows :

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathcal{P}} \arrow[r, "\iota"] \arrow[d, "h"'] & \overline{\mathcal{T}} \arrow[r, "\overline{\Theta}"] \arrow[d, "\varepsilon_1"'] & \overline{\mathcal{S}} \arrow[r] \arrow[d, "\varepsilon_2"'] & 0 \\
0 \arrow[r] & \bar{\mathfrak{p}} \arrow[r] & \bar{\mathfrak{t}} \arrow[r, "\overline{\theta}"] & \bar{\mathfrak{s}} \arrow[r] & 0
\end{tikzcd}$$

Here $\overline{\mathcal{P}} := \ker(\overline{\Theta})$, $\bar{\mathfrak{p}} := \ker(\overline{\theta})$, and h is the restriction of ε_1 on $\overline{\mathcal{P}}$. One can show that h is surjective. Because $\overline{\mathcal{M}}$ is flat over $\mathcal{M}^\varphi$ and $\mathbb{C}(M)$ is flat over $\mathbb{C}[M+M^{-1}]$,

$$\overline{\mathcal{P}} := \ker(\overline{\Theta}) = \mathcal{P} \otimes_{\mathcal{M}^\varphi} \overline{\mathcal{M}}$$

$$\bar{\mathfrak{p}} := \ker(\overline{\theta}) = \mathfrak{p} \otimes_{\mathbb{C}[M+M^{-1}]} \mathbb{C}(M).$$

Since h is surjective, and B_K is the generator of $\bar{\mathfrak{p}}$, there exists $\beta \in \overline{\mathcal{P}}$ such that $\beta|_{t=-1} = h(\beta) = B_K$. Since $\beta \in \overline{\mathcal{P}}$, one has $\alpha_K|\beta$. Thus, we have

$$(1-L) \mid \varepsilon(\alpha_K) \mid \varepsilon(\beta) = B_K, \tag{52}$$

where the fact that $(1-L) \mid \varepsilon(\alpha_K)$ is Proposition 5.12. Since the character variety has exactly 2 component,

$$B_K \overset{(M)}{=} A_K = (1-L)A'_K,$$

where the first equality means B_A is M-essentially equal to A_K. The condition (ii) implies A'_K is irreducible.

It follows that either $\varepsilon(\alpha_K) = (1-L)$, or $\varepsilon(\alpha_K) \overset{(M)}{=} (1-L)A'_K = A_K$. The first possibility can be excluded using condition (v). Hence, $\varepsilon(\alpha_K) \overset{(M)}{=} A_K$. This completes the proof of Theorem 7.3. For details, see [**LT2**].

Références

[BG] D. Bar-Natan and S. Gaorufalidis, *On the Melvin-Morton-Rozansky conjecture*, Invent. Math. 125 (1996), no. 1, 103–133.

[Bar] J. Barrett, *Skein spaces and spin structures*, Math. Proc. Camb. Phil. Soc., **126** (1999), 267–275.

[BW] F. Bonahon and H. Wong, *Representations of the Kauffman skein algebra I : invariants and miraculous cancellations*, preprint arXiv :1206.1638.

[Bul] D. Bullock, *Rings of $Sl_2(\mathbb{C})$-characters and the Kauffman bracket skein module*, Comment. Math. Helv. **72** (1997), no. 4, 521–542.

[BH] G. Brumfiel and H. Hilden, *Sl(2) representations of finitely presented groups*, Contemp. Math. **187** (1995).

[BP] D. Bullock and J. Przytycki, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc. **128** (2000), no. 3, 923–931.

[BZ] G. Burde, H. Zieschang, *Knots*, de Gruyter Stud. Math., vol. 5, de Gruyter, Berlin, 2003.

[CCGLS] D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Inventiones Math. **118** (1994), pp. 47–84.

[CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, 1994.

[CS] M. Culler and P. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. of Math. (2) **117** (1983), no. 1, 109–146.

[Du] N. Dunfield, *Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds*, Invent. Math. **136** (1999), no. 3, 623–657.

[DG] N. Dunfield and S. Garoufalidis, *Non-triviality of the A-polynomial for knots in* S^3, Algebr. Geom. Topol. **4** (2004), 1145–1153.

[FG] C. Frohman and R. Gelca, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4877–4888.

[FGL] C. Frohman, R. Gelca, and W. Lofaro, *The A-polynomial from the noncommutative viewpoint*, Trans. Amer. Math. Soc. **354** (2002), no. 2, 735–747.

[Ga1] S. Garoufalidis, *Difference and differential equations for the colored Jones function*, J. Knot Theory and Its Ramifications **17** (2008), no. 4, 495–510.

[Ga2] S. Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, 291–309 (electronic).

[GK] S. Garoufalidis and C. Koutschan, *The noncommutative A-polynomial of* $(-2, 3, n)$ *pretzel knots*, Exp. Math. **21** (2012), 241–251.

[GL1] S. Garoufalidis and T. Le, *The colored Jones function is q-holonomic*, Geom. Topol. **9** (2005), 1253–1293 (electronic).

[GL2] S. Garoufalidis and T. Le, *Asymptotics of the colored Jones function of a knot*, Geom. Topol. **15** (2011), no. 4, 2135–2180.

[Ge] R. Gelca, *On the relation between the A-polynomial and the Jones polynomial*, Proc. Amer. Math. Soc. **130** (2002), no. 4, 1235–1241.

[Ha1] K. Habiro, *On the quantum sl2 invariants of knots and integral homology spheres*, in "Invariants of knots and 3-manifolds" (Kyoto, 2001), 55–68 (electronic), Geom. Topol. Monogr., **4**, Geom. Topol. Publ., Coventry, 2002.

[Ha2] K. Habiro, *Bottom tangles and universal invariants*, Algebr. Geom. Topol. **6** (2006), 1113–1214.

[Ha3] K. Habiro, *A unified Witten-Reshetikhin-Turaev invariant for integral homology spheres*, Invent. Math. **171** (2008), no. 1, 1–81.

[Hi] K. Hikami, *Difference equation of the colored Jones polynomial for torus knot*, Internat. J. Math. **15** (2004), no. 9, 959–965.

[HL] V. Huynh and T. Le, *The colored Jones polynomial and the Kashaev invariant*, (Russian) Fundam. Prikl. Mat. 11 (2005), no. 5, 57–78 ; translation in J. Math. Sci. (N. Y.) **146** (2007), no. 1, 5490–5504.

[Hi] K. Hikami, *Difference equation of the colored Jones polynomial for torus knot*, Internat. J. Math. **15** (2004), no. 9, 959–965.

[Kas] R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275.

[KT] C. Kassel and V. Turaev, *Braid groups*, Graduate Texts in Mathemtatics **247**, Springer, 2010.

[Kau1] L. Kauffman, *States models and the Jones polynomial*, Topology, **26** (1987), 395–407.

[Kau2] L. Kauffman, *An invariant of regular isotopy*, Trans. Amer. Math. Soc. **318** (1990), 417–471.

[KM] R. Kirby and P. Melvin, The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2, C)$. *Invent. Math.* **105** (1991), no. 3, 473–545.

[Jo] V. Jones, *Polynomial invariants of knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12** (1985), 103–111.

[Le1] T. Lê , *Integrality and symmetry of quantum link invariants*, Duke Math. J. **102** (2000), 272–306.

[Le2] T. Lê, *The colored Jones polynomial and the A-polynomial of knots*, Adv. Math. **207** (2006), no. 2, 782–804.

[Le3] T. Lê, *On Kauffman bracket skein modules at root of unity*, Algebr. Geom. Topol., to appear. See also preprint arXiv :1312.3705.

[Li] W. B. R. Lickorish, *An Introduction to Knot Theory*, Springer, GTM 175, 1997.

[LM] A. Lubotzky and A. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the AMS **336** (1985).

[LT1] T. Le and A. Tran, *The Kauffman bracket skein module of two-bridge links*, Proc. Amer. Math. Soc. **142** (2014), 1045–1056.

[LT2] T. Le and A. Tran, *On the AJ conjecture for knots*, preprint arXiv :1111.5258.

[LS] W. Lück and T. Schick, *L2-torsion of hyperbolic manifolds of finite volume*, Geom. Funct. Anal. **9** (1999), 518–567.

[Mar] J. Marche, *The skein module of torus knots*, Quantum Topol. **1** (2010), no. 4, 413–421.

[Mas] G. Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, Algebr. Geom. Topol. **3** (2003), 537–556.

[Mat] T. Mattman, *The Culler-Shalen seminorms of the* $(-2,3,n)$*-pretzel knot*, J. Knot Theory Ramifications **11** (2002), no. 8, 1251–1289.

[MPL] M. Macasieb, K. Petersen and R. van Luijk, *On character varieties of two-bridge knot groups*, Proc. Lond. Math. Soc. (3) **103** (2011), no. 3, 473–507.

[MM] P. M. Melvin and H. R. Morton, *The coloured Jones function*, Comm. Math. Phys. 169 (1995), no. 3, 501–520.

[MuM] H. Murakami and J. Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, Acta Math. **186** (2001), no. 1, 85–104.

[Muk] H. Murakami, *An introduction to the volume conjecture and its generalizations*, Acta Math. Vietnam. **33** (2008), no. 3, 219–253.

[Mur] K. Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology **26** (1987), 187–194.

[Oh] T. Ohtsuki, *Quantum invariants. A study of knots, 3-manifolds, and their sets*, Series on Knots and Everything **29**, World Scientific 2002.

[Po] J. Porti, *Torsion de Reidemeister pour les variétés hyperboliques*, Mem. Amer. Math. Soc. **128** (1997), no. 612, x+139 pp.

[Pr] J. Przytycki, *Fundamentals of Kauffman bracket skein modules*, Kobe J. Math. **16** (1999) 45–66.

[PS] J. Przytycki and A. Sikora, *On the skein algebras and* $Sl_2(\mathbb{C})$*-character varieties*, Topology **39** (2000), 115–148.

[RT] N. Reshetikhin and V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Commun. Math. Phys., **127** (1990), 1–26.

[Sab] C. Sabbah *Systèmes holonomes d'équations aux q-différences*, in "D-modules and microlocal geometry" (Lisbon, 1990), 125–147, de Gruyter, Berlin, 1993.

[Sal] P. Sallenave, *Structure of the Kauffman bracket skein algebra of* $\mathbb{T}^2 \times I$, J. Knot Theory and Its Ramifications **8** (1999), no. 3, 367–372.

[Sh] I. Shafarevich, *Basic algebraic Geometry 1 : Variety in projective spave*, Springer-Verlag, 1994.

[Si] A. Sikora, *Character varieties*, Trans. Amer. Math. Soc. **364** (2012), no. 10, 5173–5208.

[Th] W. Thurston, *Three-dimensional geometry and topology*, Vol. 1 edited by Silvio Levy, Princeton Mathematical Series, 35, Princeton University Press, Princeton, NJ, 1997.

[Tr] A. Tran, *Proof of a stronger version of the AJ conjecture for torus knots*, Algebr. Geom. Topol. **13** (2013), no. 1, 609–624.

[Tu1] V. Turaev, *The Yang-Baxter equation and invariants of links*, Invent. Math. **92** (1988), 527–553.

[Tu2] V. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 6, 635–704.

[Tu3] V. Turaev, *A simple proof of the Murasugi and Kauffman theorems on alternating links*, Enseign. Math. **33** (1987), 203–225.

[Tu4] *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter & Co., Berlin, 1994.

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