

Programming Project for Math 2605

This project is aimed at developing a visual understanding of what the singular value decomposition says about the linear transformation described by a matrix A . So that we can draw graphs, we will take A to be 2×2 , so it describes a transformation from R^2 to R^2 .

Let A be any non-zero 2×2 matrix, and let $A = VDU^t$ be a singular value decomposition of A . Suppose that the rank of A is 2. Then V , U and D are all 2×2 matrices with rank 2, and hence are invertible. Since

$$V^tV = I \quad \text{and} \quad U^tU = I$$

and since U and V are invertible (as opposed to just having left inverses), they are orthogonal transformations. This means that they are either rotation or reflection matrices. The matrix D on the other hand describes a “scaling transformation”: It multiples x coordinates by a factor of σ_1 , and y coordinates by a factor of σ_2 .

Having taken A apart into a product of geometrically simple pieces, it is simple to see what the transformation described by A does to the unit circle:

(1) First, applying V^t takes the circle onto itself, since V^t is itself either a rotation matrix, or a reflection matrix.

(2) Second, applying the scale transformation D stretches – or squashes – the circle into an ellipse.

(3) Third, applying U either rotates the ellipse about its center, or reflects it about some line through the origin. Either way, the result is an ellipse with its major and minor axes running at some angle to the coordinate axes.

If A has rank 1, then the image of the unit circle is just a line segment. This is a degenerate form of ellipse; one in which the length of the minor axis is zero. The only thing to watch out for with this case is dividing by zero.

The goal here is to produce an applet, that will draw this ellipse – degenerate or not – and, in the process, determine the singular value decomposition of A .

The vectors on the unit circle are the vectors of the form

$$\mathbf{x} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}.$$

Let θ run from 0 to 2π in 1,000 evenly incremented steps. Let θ_j be the j th angle produced this way, and let \mathbf{x}_j denote $\begin{bmatrix} \cos(\theta_j) \\ \sin(\theta_j) \end{bmatrix}$.

Compute $A\mathbf{x}_j$ for each j . These are points on the ellipse that we seek to draw. Before drawing it, we find the major and minor axes of the ellipse. To find these, determine a value j_{\max} of j that makes $|A\mathbf{x}_j|$ largest, and determine a value j_{\min} of j that makes $|A\mathbf{x}_j|$ smallest. This tells us the singular values of A :

$$\sigma_1 = |A\mathbf{x}_{j_{\max}}| \quad \text{and} \quad \sigma_2 = |A\mathbf{x}_{j_{\min}}|.$$

(We say *a* value instead of *the* values since if you are working exactly, there would be two angles, π apart, that would give the same length – corresponding to the opposite ends of both the major and minor axes on the ellipse. Probably because of rounding effects, one or the other will lead to a slightly larger result in your numerical computation).

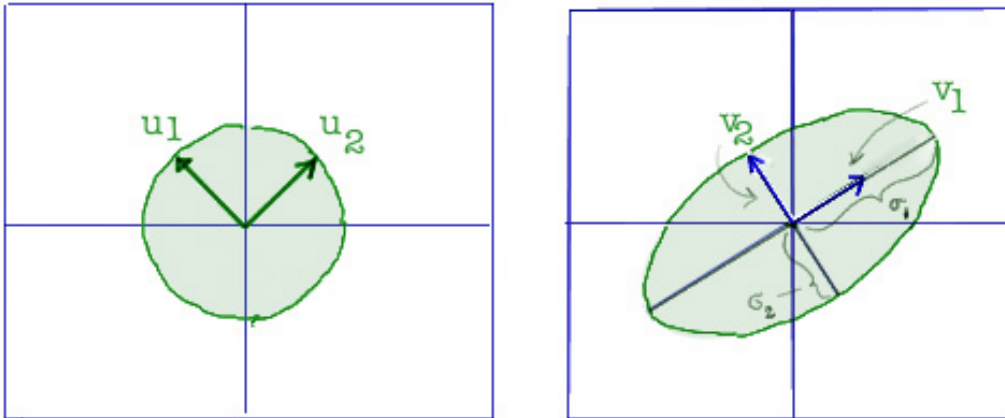
It also tells us the vectors \mathbf{u}_1 , \mathbf{u}_2 , \mathbf{v}_1 and \mathbf{v}_2 : We have

$$\mathbf{u}_1 = \mathbf{x}_{j_{\max}} \quad \text{and} \quad \mathbf{u}_2 = \mathbf{x}_{j_{\min}}$$

while

$$\mathbf{v}_1 = \frac{1}{|A\mathbf{x}_{j_{\max}}|} A\mathbf{x}_{j_{\max}} \quad \text{and} \quad \mathbf{v}_2 = \frac{1}{|A\mathbf{x}_{j_{\min}}|} A\mathbf{x}_{j_{\min}} .$$

The idea is to have a graph with two panels: one for the input variables, and one for the output variables. On the left panel, show the unit circle and the vectors \mathbf{u}_1 and \mathbf{u}_2 . On the right panel, draw the ellipse and the vectors \mathbf{v}_1 and \mathbf{v}_2 .



Below the graph, show the original matrix, and give the singular value decomposition of A . To specify D , you just need to specify σ_1 and σ_2 . To specify U and V recall that the general 2×2 rotation matrix is

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} ,$$

and the general 2×2 rotation matrix is

$$M(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} .$$

We will have either $U = R(\theta_{\max})$ or $U = M(\theta_{\max})$ depending on whether $\det(U) = 1$ or $\det(U) = -1$. This you can easily check, so you can report something like:

$$U = R(\theta) \text{ where } \theta = 1.213$$

and you can do something similar for V except now you have to compute the angle θ , which is given by $\mathbf{v}_1 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$.

The applet should also have four text fields for entering the matrix A . For extra credit, make the graphs in the applet interactive: Put a red dot on the circle in the input graph, and make it so that it can be dragged around the circle with the mouse. On the output side – the graph of the ellipse – show the vector corresponding to the image of this movable input vector. This allows the maximization and minimization procedure to be visualized. Otherwise, it just runs in the background.