Solutions of selected problems of chapter 1

Problem 1.2

It is clear the function $f(x,y)$ is continuous at all $(x,y) \neq (0,0)$. Thus it remains to show that it is continuous at $(x,y) = (0,0)$. For any $\varepsilon > 0$ we have to find δ so that $|f(x,y)| < \varepsilon$ whenever $|\mathbf{x}| = \sqrt{x^2+y^2} < \delta$. To this we have to estimate $|f(x,y)|$ in terms of $\sqrt{x^2+y^2}$. Since $|x+y| = |x \cdot 1 + y \cdot 1| \leq \sqrt{2}\sqrt{x^2+y^2}$, by Schwarz's inequality we have that

$$|f(x,y)| \leq 2\sqrt{2}|\mathbf{x}| \ln(|\mathbf{x}|) \ .$$

We know from calculus in one variable that the logarithm function tends towards $-\infty$ very slowly so that

$$t^r \ln t$$

tends to zero as $t \to 0$ for all $r > 0$. Pick any $0 < r < 1$, say $r = 1/2$ and note that for $0 < t \leq 1$ we have that

$$t^{1/2} \ln t \leq 2e \ .$$

This follows by finding the maximum of the function $t^{1/2} \ln t$ on the interval $(0.1]$. Hence we have that

$$|f(x,y)| \leq 2e|\mathbf{x}|^{1/2}$$

and if we choose

$$\delta = \frac{\varepsilon}{2e}^{\ 2}$$

we learn that whenever $|\mathbf{x}| < \delta$ then $|f(x,y)| < \varepsilon$.

Notice that there was nothing magic about the power $1/2$; any positive power that is strictly smaller than one would have done the job, maybe not in such a simple fashion.

Problem 1.3

For any given ε pick

$$\delta = \varepsilon/M$$

and then note that

$$|f(\mathbf{x}) - f(\mathbf{y}) \leq M|\mathbf{x} - \mathbf{y}| < \varepsilon$$

whenever

$$|\mathbf{x} - \mathbf{y}| < \delta = \varepsilon/M \ .$$

Problem 1. 5

Note that from the fundamental theorem of calculus

$$\sin(x) - \sin(y) = \int_y^x \cos(s) ds$$

and hence

$$|\sin(x) - \sin(y)| \le |x - y| \ .$$

Similarly we also have that

$$|\cos(x) - \cos(y)| \le |x - y| \ .$$

Hence, using the formula 1.16 in the statement of the problem we get that

$$|f(x_1, x_2) - f(y_1, y_2)| \le |\cos(x_2)||x_1 - -y_1| + |\sin(y_1)||x_2 - y_2| \le |x_1 - -y_1| + |x_2 - y_2| \ .$$

Now, as in problem 2 we write

$$|x_1 - -y_1| + |x_2 - y_2| = |x_1 - -y_1| \cdot 1 + |x_2 - y_2| \cdot 1$$

and apply Schwarz's inequality to obtain

$$|x_1 - -y_1| + |x_2 - y_2| \le \sqrt{2}|\mathbf{x} - \mathbf{y}| \ .$$

Combining this with the other estimates we get that

$$|f(\mathbf{x}) - f(\mathbf{y})| \le \sqrt{2}|\mathbf{x} - \mathbf{y}| \ .$$

Problem 1.6

The function $\mathbf{F}(\mathbf{x})$ is given by

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ \cdot \\ \cdot \\ \cdot \\ f_m(\mathbf{x}) \end{bmatrix} ,$$

and each of the functions $f_j(\mathbf{x})$ is continuous.

This means that for $\varepsilon > 0$ there exists δ_j (one for each function) so that $|f_j(\mathbf{x}) - f_j(\mathbf{y})| < \varepsilon/\sqrt{m}$ whenever $|\mathbf{x} - \mathbf{y}| < \delta_j$. Now

$$|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| = \sqrt{\sum_{j=1}^{m} |f_j(\mathbf{x}) - f_j(\mathbf{y})|^2}$$

and if we choose $\delta = \min\{\delta_j\}$ we get that

$$|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})| < \sqrt{\sum_{j=1}^{m} \varepsilon^2/m} = \varepsilon$$

whenver $|\mathbf{x} - \mathbf{y}| < \delta$.

Problem 1.7

It follows from the continuity of functions in one variable that the function $f(x, y)$ is continuous at $(x, y) \neq (0, 0)$. The tricky part is to show the continuity at $(0, 0)$. It is at this step where the condition $r < 2$ plays a role. As always we need an estimate of $f(x, y)$ in terms of $|\mathbf{x}|$.

A useful inequality is the one between the arithmetic and geometric mean

$$2|x||y| \leq x^2 + y^2 \ ,$$

which follows from the fact that $(|x| - |y|)^2 \geq 0$.

Write

$$2|x||y| = (|x||y|)^{1-r/2} 2|x|^{r/2}|y|^{r/2}$$

which by the arithmetic geometric mean inequality is bounded above by

$$(|x||y|)^{1-r/2}(|x|^r + |y|^r) \ .$$

Hence

$$|f(x, y)| \leq (|x||y|)^{1-r/2}$$

which, again by the arithmetic geometric mean inequality can be estimated by

$$|f(x, y)| \leq (|x||y|)^{1-r/2} \leq \left[\frac{1}{2}(x^2 + y^2)\right]^{1-r/2} \ ,$$

i.e.,

$$|f(x, y)| \leq \left[\frac{1}{2}\right]^{1-r/2} |\mathbf{x}|^{1-r/2} \ .$$

from this the continuity at $(0, 0)$ follows in a routine fashion.

If $r \geq 2$, then the function is not continuous. Fot $t \neq 0$ consider

$$f(t, t) = |t|^{2-r} \ ,$$

which does not tend to zero as $t \to 0$.

Problem 2.1

$$\nabla f(x, y) = (4x + y, x + 2y) \ .$$

$$\nabla h(x, y) = \frac{1}{2\sqrt{f(x, y)}} \nabla f(x, y) \ .$$

$$\nabla f(1, 1) = (5, 3) \ ,$$

$$\nabla h(1,1) = \frac{1}{4}(5,3) \ .$$

Problem 2.3

$$\nabla f(x,y) = (\cos(x^2 y) - 2x^2 y \sin(x^2 y), -x^3 \sin(x^2 y)) \ .$$

$$\nabla g(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x,y) \ .$$

Using these two results the gradient of $f \cdot g$ can be computed by the product rule:

$$\nabla (f \cdot g)(x,y) = \nabla f(x,y) g(x,y) + f(x,y) \nabla g(x,y) \ .$$

Problem 2.5

One has to show that the limits

$$\lim_{t \to 0} \frac{f(x+t,y) - f(x,y)}{t} \ , \lim_{t \to 0} \frac{f(x,y+t) - f(x,y)}{t}$$

both exist. Now by the definition of the derivative in one variable

$$g(h(x+t,y)) - g(h(x,y)) = g'(h(x,y))(h(x+t,y) - h(x,y)) + o(1) \cdot (h(x+t,y) - h(x,y))$$

where $o(1)$ tends to zero as $|h(x+t,y) - h(x,y)| \to 0$. Next,

$$\frac{g(h(x+t,y)) - g(h(x,y)}{t} = g'(h(x,y)) \frac{(h(x+t,y) - h(x,y))}{t} +$$

$$o(1) \cdot \frac{(h(x+t,y) - h(x,y))}{t} \ ,$$

and as $t \to 0$ the right side converges to

$$g'(h(x,y)) \frac{\partial h}{\partial x}(x,y)$$

since $o(1) \to 0$ as $t \to 0$. The argument for the partial derivative with respect to y is the same. Thus, we get that with $f(x,y) = g(h(x,y))$

$$\nabla f(x,y) = g'(h(x,y)) \nabla h(x,y) \ .$$

Problem 2.7

The gradient always points in the direction of steepest increase of the function. This the direction of steepest it decrease is the opposite direction, i.e., the negative of the gradient. In the case at hand

$$\nabla f(-1,2) = \frac{2}{9}(5,-4)$$

and hence the direction is

$$-\frac{2}{9}(5,-4)\ .$$

Problem 2.9

$$\mathbf{x}_0 + t\mathbf{v} = (1+2t, 2+t)\ ,$$

and hence

$$g(t) = 2(1+2t)^2 + (1+2t)(2+t)^3 + (2+t)^2\ .$$

The derivative at $t=1$ has the value 165. Next

$$\frac{d}{dt} f(\mathbf{x}_0 + t\mathbf{v}) = \nabla f(\mathbf{x}_0 + t\mathbf{v}) \cdot \mathbf{v}$$

which has to be evaluated at $t=1$. The gradient of f is

$$(4x + y^3, 3xy^2 + 2y)\ ,$$

and at $\mathbf{x}_0 + \mathbf{v} = (3,3)$ it is

$$(39, 87)$$

and $(39,87) \cdot (2,1) = 78 + 87 = 165$.