Section 5:

Problem 1 Think of the problem in the following way. A three by three unitary matrix that has the vector $\mathbf{u}$ as its first column maps the vector $\mathbf{e_1}$ to the vector $\mathbf{u}$. Thus it suffices to find a Householder reflection that maps the vector $\mathbf{u}$ to $\mathbf{e_1}$. This Householder reflection is given by

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{\sqrt{3}-1}{2} & -\frac{\sqrt{3}+1}{2} \\ 1 & -\frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \end{bmatrix}$$

Thus, the second and third column vectors form a basis for the plane given by the equation $x + y + z = 0$.

Problem 3 This is the same as problem 1 in section 1.

Problem 5 The matrix A is symmetric and hence can be diagonalized. It has the obvious eigenvalue 6 with the normalized eigenvector

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

The characteristic polynomial of the matrix A is given by

$$t^3 - 4t^2 - 12t$$

and has the roots $6, 0, -2$. The normalized eigenvector associated with the eigenvalue 0 is

$$\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The normalized eigenvector associated with the eigenvalue -4 is given by

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

Hence the Schur factorization is given by $A = QDQ^t$ where

$$D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Section 6:

Problem 1

$$e^tT = \begin{bmatrix} e^{3t} & 2(e^{3t} - e^{2t}) \\ 0 & e^{2t} \end{bmatrix} .$$

Problem 3 Quite generally, consider a matrix of the form

$$T = \mu I + U$$

where N has the property that $U^2 = 0$. An example for such a matrix is

$$U = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} .$$

One calculates easily that

$$T^2 = \mu^2 I + 2\mu U + U^2 = \mu^2 I + 2\mu U$$

since $U^2 = 0$. Assume now that

$$T^{k-1} = \mu^{k-1} I + (k-1)\mu^{k-2} U$$

and calculate

$$\begin{aligned} T^k &= TT^{k-1} = [\mu I + U]\left[\mu^{k-1} I + (k-1)\mu^{k-2} U\right] \\ &= \mu^k I + (k-1)\mu^{k-1} U + \mu^{k-1} U + \mu^{k-2} U^2 = \mu^k I + k\mu^{k-1} U \ , \end{aligned}$$

since $U^2 = 0$. Hence, by mathematical induction we have proved that

$$T^k = \mu^k I + k\mu^{k-1} U \ ,$$

for all $k = 0, 1, 2, \ldots$.

Next,

$$\begin{aligned} e^{tT} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} T^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^k I + \sum_{k=0}^{\infty} \frac{t^k}{k!} k\mu^{k-1} U \\ &= e^{\mu t} I + \left[\sum_{k=0}^{\infty} \frac{t^k}{(k-1)!} \mu^{k-1}\right] U \end{aligned}$$

The term in parenthesis equals

$$t\left[\sum_{k=0}^{\infty} \frac{t^{k-1}}{(k-1)!} \mu^{k-1}\right] = te^{\mu t} .$$

hence we learn that

$$e^{tT} = e^{\mu t}\left[I + tU\right] .$$

For the problem at hand we get

$$e^{tT} = e^{2t}\begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} .$$

Problem 5

a) The eigenvalues of the matrix A are

$$\mu_1 = 5 + 4i \ , \mu_2 = 5 - 4i \ .$$

The normalized eigenvector associated with the eigenvalue μ_1 is given by

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

and unitary matrix Q that will effect the Schur factorization is

$$Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 2i & 1 \\ 1 & 2i \end{bmatrix} .$$

Clearly

$$Q^*AQ = \frac{1}{\sqrt{5}}\begin{bmatrix} -2i & 1 \\ 1 & -2i \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -8+10i & 5-16i \\ 5+4i & 2+10i \end{bmatrix} = \begin{bmatrix} 5+4i & -6 \\ 0 & 5-4i \end{bmatrix} .$$

b) Next we compute

$$e^{tT} = \begin{bmatrix} e^{(5+4i)t} & a \\ 0 & e^{(5-4i)t} \end{bmatrix} ,$$

where a is unknown. Since

$$Te^{tT} = e^{tT}T \ .$$

we get for a the equation

$$(5+4i)a - 6e^{(5-4i)t} = -6e^{(5+4i)t} + a(5-4i) \ ,$$

and hence

$$a = \frac{3\left(e^{(5-4i)t} - e^{(5+4i)t}\right)}{4i} .$$

Thus

$$e^{tT} = \begin{bmatrix} e^{(5+4i)t} & \frac{3\left(e^{(5-4i)t}-e^{(5+4i)t}\right)}{4i} \\ 0 & e^{(5-4i)t} \end{bmatrix} .$$

c) Finally,

$$\begin{aligned} e^{At} = Qe^{tT}Q^* &= \frac{1}{\sqrt{5}}\begin{bmatrix} 2i & 1 \\ 1 & 2i \end{bmatrix}\begin{bmatrix} e^{(5+4i)t} & \frac{3\left(e^{(5-4i)t}-e^{(5+4i)t}\right)}{4i} \\ 0 & e^{(5-4i)t} \end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix} -2i & 1 \\ 1 & -2i \end{bmatrix} \\ &= \begin{bmatrix} \frac{A+B}{2} & i(A-B) \\ -i\frac{A-B}{4} & \frac{A+B}{2} \end{bmatrix} \end{aligned}$$

where $A = e^{(5+4i)t}$ and $B = e^{(5-4i)t}$. Using Euler's fromula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ this can be simplified and yields

$$e^{tA} = \begin{bmatrix} e^{5t}\cos(4t) & -2e^{5t}\sin(4t) \\ \frac{1}{2}e^{5t}\sin(4t) & e^{5t}\cos(4t) \end{bmatrix} .$$