Solutions of selected problems of chapter 5

Section 1:

Problem 1: a)

$$\mathbf{v}(t) = r \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} \ , \ \mathbf{a}(t) = -r \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \ .$$

b) $v(t) = r$, $\mathbf{T}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}$.
c) $s(t) = rt$, $t(s) = s/r$.
d) The tangential component of the acceleration is zero. The normal component is r.
e) The curvature is $1/r$.

Problem 3: a)

$$\mathbf{v}(t) = \begin{bmatrix} 1 \\ 1/\sqrt{t} \\ -1/t^2 \end{bmatrix} \ , \ \mathbf{a}(t) = \begin{bmatrix} 0 \\ -1/(2t^{3/2}) \\ 2/t^3 \end{bmatrix} \ .$$

b) $v(t) = \sqrt{1 + 1/t + 1/t^4}$ and

$$\mathbf{T}(t) = \frac{1}{\sqrt{1 + 1/t + 1/t^4}} \begin{bmatrix} 1 \\ 1/\sqrt{t} \\ -1/t^2 \end{bmatrix} \ .$$

c) $s(t) = \int \sqrt{1 + 1/t + 1/t^4} dt$
d)

$$\mathbf{a}(t) = \mathbf{a}(t) \cdot \mathbf{T}(t)\mathbf{T}(t) + [\mathbf{a}(t) - \mathbf{a}(t) \cdot \mathbf{T}(t)\mathbf{T}(t)] \ ,$$

The first term yields the tangential component and the econd term the normal component. The tangential component is given by

$$-\frac{t^3/2 + 2}{t^5 + t^4 + t} \begin{bmatrix} 1 \\ 1/\sqrt{t} \\ -1/t^2 \end{bmatrix} \ .$$

e)

$$\kappa(t) =$$

Section 2:

Problem 1:

$$x'' = -x^2 + 2xx' - 2x \ ,$$

yields

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ -x^2 + 2xy - 2x \end{bmatrix}$$

Problem 3:
With $x' = y$, $y' = \frac{(m^2-t^2)x-ty}{t^2}$ and hence

$$\mathbf{F}(x,t) = \begin{bmatrix} y \\ \frac{(m^2-t^2)x-ty}{t^2} \end{bmatrix} .$$

Problem 5:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

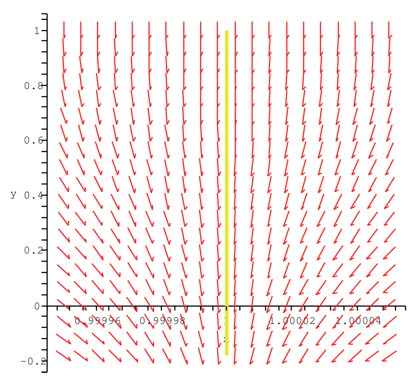
and hence

$$\begin{aligned} e^{At} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{4t} + e^{-2t} & e^{4t} - e^{-2t} \\ e^{4t} - e^{-2t} & e^{4t} + e^{-2t} \end{bmatrix} . \end{aligned}$$

Finally, the solution for the initial value problem is given by

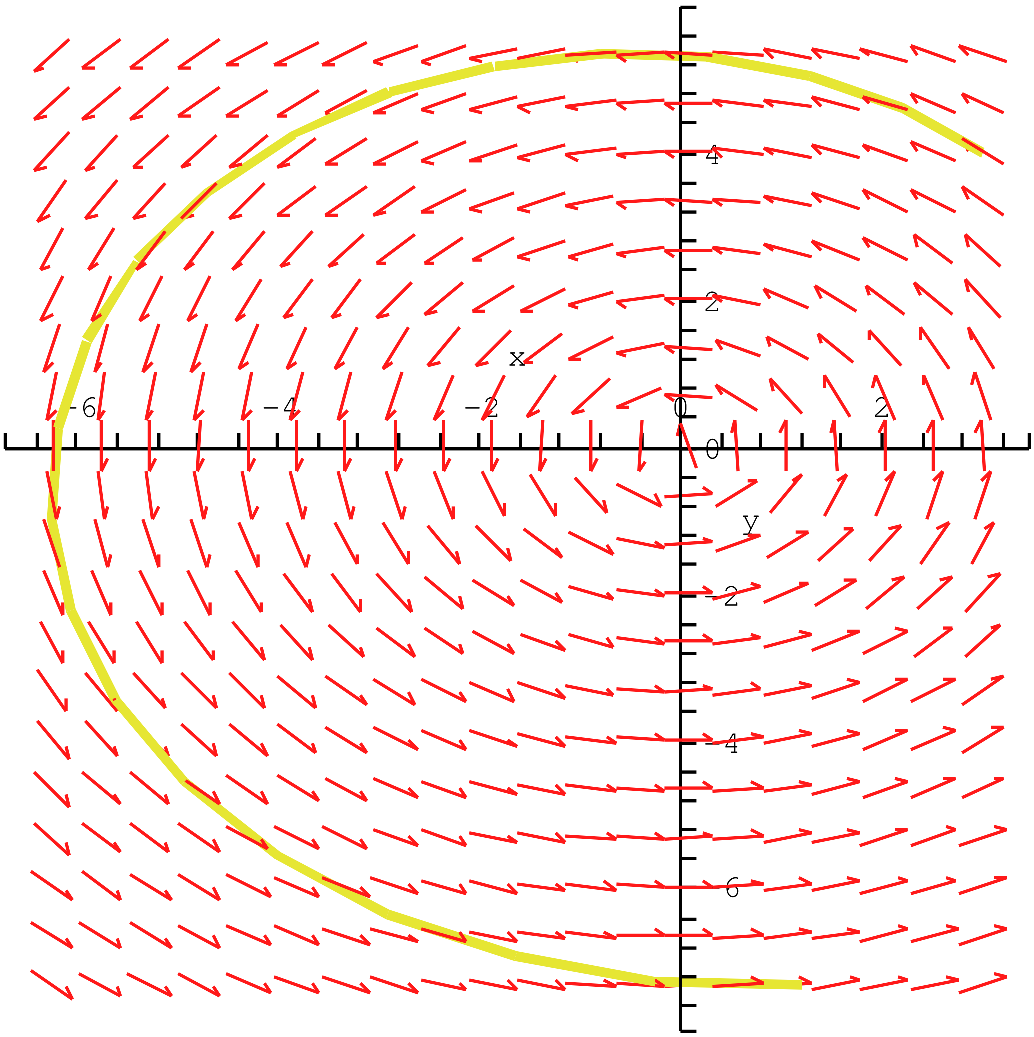
$$\mathbf{x}(t) = e^{At} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3e^{4t} - e^{-2t} \\ 3e^{4t} + e^{-2t} \end{bmatrix}$$

Problem 7:



This problem can, in principle, be solved exactly by working out integrals. Noting that the variable $x(t)$ stays equal to one for all times, i.e., $x(t) = 1$, we have to solve $y' = -y^3 - 1$ which can be done by separation of variables.

Problem 9: a)



b) The Euler scheme leads to

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\mathbf{x}_n^\perp \ ,$$

and hence

$$|\mathbf{x}_{n+1}|^2 = |\mathbf{x}_n + h\mathbf{x}_n^\perp|^2 = |\mathbf{x}_n|^2(1 + h^2) \ .$$

In the case at hand

$$|\mathbf{x}_{20}| = |\mathbf{x}_0|^{10}(1 + 0.04)^{10} \approx 7.4 \ .$$

c) Since $Nh \approx 2\pi$ and apparent closure requires that

$$(1+h^2)^N \approx 1.05$$

we must have that

$$(1+(\frac{2\pi}{N})^2)^N \approx 1.05 \ .$$

Calculate, e.g.,

$$5 * (1+(6.28/4000)^2)^{4000} \approx 5.049543790$$

Thus we need N to be about 4000, or h to be about $2\pi/4000$.

Section 3:

Problem 1: Equilibrium points:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} , \begin{bmatrix} -1 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

The Jacobi matrices are: at $(0,0)$

$$\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} ,$$

at $(1,-1)$ and $(-1,1)$

$$\begin{bmatrix} -4 & 0 \\ -1 & -3 \end{bmatrix} .$$

The determinant of the first is -1 and hence there is a positive and negative eigenvalue and the equilibrium point is unstable. For the others, the eigenvalues are -4 and -3 and hence these two equilibrium points are stable.

Problem 3: The equilibrium points are

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} , \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

The Jacobi matrices are: at $(0,0)$

$$\begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} ,$$

at $(1,1)$

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} ,$$

and at $(1,1/2)$

$$\begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix} .$$

In the first case, nothing can be said about the stability or instability, because one of the eigenvalues vanishes. I n the second case the eigenvalues are $1/2 \pm i\sqrt{7}/2$ and the equilibrium point is unstable. The last has the eigenvalue -1 and $1/2$ and, again, is unstable.