Prepfinal A for Calculus III for CS-Majors, Math 2605A1-2
April 24, 2003

Name:

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. You may use a 'cheat sheet' of 1 page, single sided, letter format. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

Block 1:

I: Two surfaces are given by the equations $z = x^3 + 2y$ and $z = 4x^2 - y^2$. Find the line tangent to the intersection of the two surfaces at the point $(1, 1, 3)$. Give this line in parametrized form.

II: Find all the points in the domain $(x - 2)^2 + y^2 \leq 1$ where the maxima and minima of the function $f(x, y) = \log(x^2 + y^2)$. To do this, find all the critical points in the interior of the unit disk and analyze the Hessian. Then maximize this function on the boundary of the unit disk. Sketch a few level curves of this function.

III: Find all the critical points of the function

$$f(x, y) = \frac{x^2 - y^2}{(1 + x^2 + y^2)^2}$$

and discuss them by analyzing the Hessian. Draw a few level curves of this function.

IV: Find a solution of the system of nonlinear equations

$$x + y^3 = 3 \ , x^2 + 2y^2 = 4 \ ,$$

using Newton's method, starting from the point $(1, 1)$. Run two steps of the iteration and plug the approximate solution into the original equation to see how precise it is.

Block 2:

V: Diagonalize, as well as find the Schur decomposition of the matrix

$$\begin{bmatrix} -2 & 2 \\ 8 & 4 \end{bmatrix} .$$

VI: a) Using Householder reflections, find the QR factorization of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

b) Find a least square solution for the equation $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} .$$

VII: a) Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 0 & \sqrt{3} \\ 1 & 2 \\ \sqrt{2} & \sqrt{2} \end{bmatrix} .$$

b) Use this to find the least square approximation of smallest length of the equation $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} -\sqrt{3} \\ 3 \\ 0 \end{bmatrix}$$

c) Find the best rank one approximation $A_{(1)}$ for the matrix A.

VIII: Find the Housholder reflection that maps the vector

$$\begin{bmatrix} i \\ 1 \\ 1+i \end{bmatrix}$$

to a multiple of $\vec{e}_1$.

Block 3:

IX: a) Consider the function $f(x, y) = 1-((x/\sqrt{2})^2+y^2)$. Consider the surface $z = f(x, y)$ as a mountain. Suppose you start at the foot of the mountain, at the point $(1/\sqrt{2}, \sqrt{3}/2)$ and walk up the mountain on a path that points always in the direction of steepest ascent.

a) Give this path in parametrized form.

b) Find an equation for this path.

c) Sketch this path.

X: a) Find the axis $\vec{e}$ and the angle of rotation $0 \leq \phi < \pi$ of the rotation

$$\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{bmatrix} .$$

b) Find the matrix $e^{B_{\vec{e}}\theta}$ for all $0 \leq \theta \leq 2\pi$.

XI: Let Ω be the parallelogram bounded by $x + y = 0$, $x + y = 1$, $x - y = 0$, $x - y = 2$. Evaluate

$$\int_\Omega (x^2 + y^2) dx dy .$$

XII: Compute the volume of the set that is bounded above by the plane $z = 2x$ and below by the disk $(x - 1)^2 + y^2 \leq 1$.