Solutions for Prepfinal A

I:

Tangent planes: $3x + 2y - z = 2$and $8x - 2y - z = 3$. Line of intersection:

$$x = 1 - 4t \ , y = 1 - 5t \ , z = 3 - 22t \ .$$

II: $\log(x^2 + y^2)$ has no critical point inside $(x - 2)^2 + y^2 < 1$. Hence the maxima and minima are on the boundary $(x - 2)^2 + y^2 = 1$. The max is at the point $(3, 0)$ and the value is $2 \log 3$. The minimum is at the point $(1, 0)$ and the min is 0.

III: Critical points: $(0, 0), (0, \pm 1), (\pm 1, 0)$. Hessian at $(0, 0)$:

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \ , \text{ saddle}$$

at $(0, \pm 1)$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \ , \text{ local min}$$

at $(\pm 1, 0)$

$$\begin{bmatrix} -1/2 & 0 \\ 0 & -1 \end{bmatrix} \ . \text{ local max}$$

IV: If $\vec{x}_0$ is the initial value then the first approximant is

$$\vec{x}_1 = \vec{x}_0 - J_{\vec{f}}^{-1}(\vec{x}_0)\vec{f}(\vec{x}_0)$$

which leads to

$$\vec{x}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \ ,$$

and

$$\vec{x}_2 = \begin{bmatrix} 3/4 \\ 4/3 \end{bmatrix} \ .$$

Plugging this into the equations yields the values $13/108$ and $-17/144$.

V: Eigenvalues are $6, -4$. Eigenvectors are the column vectors of the matrix

$$V = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$$

Schur decomposition: Set

$$U = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix}$$

Note that the first column vector is the eigenvector with eigenvalue 6 and the second vector is perpendicular to this eigenvector. Both are normalized. Now

$$U^*AU = \frac{1}{17}\begin{bmatrix} 102 & 12 \\ 0 & -68 \end{bmatrix}$$

which is upper triangular.

VI: Householder reflections:

$$M_1 = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & 0 \end{bmatrix}$$

and

$$M_2 = \frac{1}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -3 \\ 0 & -3 & 4 \end{bmatrix} .$$

$$A = M_1 M_2 \begin{bmatrix} 9 & 8 & 1 \\ 0 & 5 & -\frac{19}{5} \\ 0 & 0 & -\frac{8}{5} \end{bmatrix} .$$

VII: $A = VDU^T$ where

$$V = \frac{1}{\sqrt{55}}\begin{bmatrix} 2\sqrt{3} & \sqrt{3}\sqrt{11} \\ 5 & 0 \\ 3\sqrt{2} & -\sqrt{2}\sqrt{11} \end{bmatrix}, D = \begin{bmatrix} \sqrt{11} & 0 \\ 0 & 1 \end{bmatrix}, U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

VIII: The Householder reflection

$$I - 2\frac{1}{6}\begin{bmatrix} 1 & -i & -1-i \\ i & 1 & 1-i \\ -1+i & 1+i & 2 \end{bmatrix}$$

maps the given vector to the vector $(2i, 0, 0)$.

IX: Think of the curve given in parametrized form, i.e., $x(t), y(t), z(t)$ where $z(t) = 1-((x(t)/\sqrt{2})^2 + y(t)^2)$. Moving in the direction of steepest ascent means that the velocity points in the direction of the gradient, i.e.,

$$\dot{x} = -x \ , \dot{y} = -2y \ .$$

thiis is a system of differential equations which we have to solve together with the initial conditions $x(0) = 1/\sqrt{2}, y(0) = \sqrt{3}/2$. The solutions are

$$x(t) = e^{-t}1/\sqrt{2} \ , y(t) = e^{-2t}\sqrt{3}/2 \ ,$$

which, together with $z(t)$ yields the curve.

The x and y components of this curve satisfy the equation

$$(\sqrt{2}x)^2 = 2y/\sqrt{3} \; .$$

X: a) The axis of rotation is

$$\vec{e} = -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and the angle is $\arccos(-1/3)$.

b) $e^{B_{\vec{e}}\theta}$ is given by Euler's formula

$$\cos(\theta)I + (1-\cos(\theta))\vec{e}\vec{e}^T + \sin(\theta)B_{\vec{e}} = \begin{bmatrix} \frac{1+\cos(\theta)}{2} & \frac{1-\cos(\theta)}{2} & -\sin(\theta)/\sqrt{2} \\ \frac{1-\cos(\theta)}{2} & \frac{1+\cos(\theta)}{2} & \sin(\theta)/\sqrt{2} \\ \sin(\theta)/\sqrt{2} & -\sin(\theta)/\sqrt{2} & \cos(\theta) \end{bmatrix}$$

XI: In the new variables

$$u = x + y \; , v = x - y$$

the integral is given by

$$\frac{1}{4}\int_0^1 \int_0^2 (u^2 + v^2) dv du = \frac{5}{6}$$

XII: The volume is 2π.