Prepfinal C for Calculus III for CS-Majors, Math 2605A1-2
April 24, 2003

Name:

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. You may use a 'cheat sheet' of 1 page, single sided, letter format. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

Block 1:

I: Given a function $f(x, y, z) = x^2y + y^2z + z^2x$.

a) Find the gradient and the Hessian of this function.

b) Find the plane tangent to the level surface $f(x, y, z) = 3$ at the point $(1, 1, 1)$.

c) Find all the critical points for this function.

II: Find all the points in the domain $x^2 + y^2 + z^2 \leq 1$ where the function $x^4 + y^4 + z^4$ attains its maximum value and minimum value and calculate these values.

III: Find all the critical points of the function

$$f(x, y) = \frac{xy}{(1 + x^2 + y^2)^2}$$

and discuss them by analyzing the Hessian. Draw a few level curves of this function.

IV: Find a solution of the system of nonlinear equations

$$x + 2y^3 = -3\sqrt{2}\ ,\ x^2 + y^2 = 4\ ,$$

using Newton's method, starting from the point $(1, -1)$. Run one step of the iteration and plug the approximate solution into the original equation to see how precise it is.

Block 2:

V: Diagonalize, as well as find the Schur decomposition of the matrix

$$\begin{bmatrix} 7 & 5 \\ -5 & 1 \end{bmatrix}\ .$$

VI: a) Using Householder reflections, find the QR factorization of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 4 \\ 0 & 3 \end{bmatrix}$$

b) Find a least square solution for the equation $A\vec{x} = \vec{b}$ where

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

VII: A matrix A has the singular value decomposition $A = VDU^T$ where

$$V = \begin{bmatrix} 3/\sqrt{17} & 2/3 \\ 2/\sqrt{17} & -1/3 \\ 2/\sqrt{17} & -2/3 \end{bmatrix} , D = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix} , U = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} .$$

Find the lowest rank approximation $A_{(1)}$.

VIII: Compute e^{At} where

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} .$$

Block 3:

IX: Consider the system of differential equations

$$\dot{x} = y + \cos(x) - 1 \ , \dot{y} = -\sin(x) \ .$$

a) Find all the critical (equilibrium) points of this system.

b) Linearize the system in the vicinity of these critical points.

c) What types of critical points do these linear systems have?

d) What are the types of critical points that the nonlinear system might have?

e) Which ones are stabel and which ones are unstable.

X: Solve the initial value problem

$$\dot{x} = -2x + 2y \ , \dot{y} = 8x = 4y$$

with initial conditions $x(0) = 1, y(0) = 2$.

XI: Consider the curve

$$x = \cos(t) \ , y = \sin(t) \ , z = 2t \ ,$$

with $0 \leq t \leq 2\pi$.

a) Find the length of this curve.

b) Rewrite the curve in the length parametrization s.

c) In this new parametrization calculate the unit tangent vector $\vec{T}(s)$, the normal vector $\vec{N}(s)$ and the binormal vector $\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$.

d) Find the curvature and the torsion of this curve.

XII: Find the integral of the function x^2y over the set Ω that is bounded by the curves $xy = 1$, $xy = 4$, $y = x$ and $y = 4x$.