Solutions to Practice Test 1A

1) a) $\nabla f(\vec{x}) = 4(x^3-y, y^3-x)$

$\vec{x}_0 = (2^{1/4}, 0)$

$\nabla f(\vec{x}_0) = 4(2^{3/4}, -2^{1/4})$.

b) $2^{3/4}x - 2^{1/4}y = 2$.

c) Tangent line parallel to x-axis means that the gradient is perpendicular to the x-axis, i.e.,

$x^3 - y = 0 \qquad f(\vec{x}_0) = 2$

$x^4 + y^4 - 4xy = 2$

Eliminate y: $x^{12} - 3x^4 = 2$.

Set $z = x^4$: $z^3 - 3z = 2$.

Solutions: $z_1 = -1$ (double root)

$z_2 = 2$

Since $z = x^4 \geq 0$ the positive root is of interest (the neg. not!)

$x = \pm 2^{1/4} \quad y = \pm 2^{3/4}$

$\vec{x}_1 = (2^{1/4}, 2^{3/4})$, $\vec{x}_2 = -(2^{1/4}, 2^{3/4})$ are the points.

d) The curve is symmetric under the exchange of x and y

$\Rightarrow \vec{x}_3 = (2^{3/4}, 2^{1/4})$, $\vec{x}_4 = -(2^{3/4}, 2^{1/4})$

are the points where the tangent is vertical.

e) Tangent line parallel to the $x=y$ axis; i.e. gradient perp. to the $x=y$ axis.

$x^3 - y = -(y^3 - x)$ or $\underbrace{(x^3 - x) = -(y^3 - y)}_{\text{I}}$ and

$\underbrace{x^4 + y^4 - 4xy = 2}_{\text{II}}$

I can be written as

$(x^3 + y^3) - (x+y) = (x+y)\cdot[x^2 - xy + y^2 - 1] = 0$

a) $x+y=0 \Rightarrow x=-y$ From II

$x^4 + x^4 + 4x^2 = 2$ or $x^4 + 2x^2 = 1$

$z = x^2 \Rightarrow z^2 + 2z = 1$ or $z_1 = \sqrt{2} - 1$

$z_2 = -\sqrt{2} - 1$

only $z_1 > 0$ counts. The solutions are

$x = \pm\sqrt{\sqrt{2} - 1}$

$y = \mp\sqrt{\sqrt{2} - 1}$

b) $x^2+y^2-xy-1=0$ From this we get

$$(x^2+y^2)^2 = (1+xy)^2$$
$$\|$$
$$x^4+y^4+2x^2y^2 = 1+2xy+x^2y^2 \text{ or}$$

$$x^4+y^4 = 1+2xy-x^2y^2 \quad (*)$$

Inserting this into II yields

$$1+2xy-x^2y^2-4xy = 2 \quad \text{or with } z=xy$$

$$z^2+2z+1=0 \quad \text{or } (z+1)^2=0 \quad \text{i.e. } \underline{z=-1}$$

Thus $xy=-1$ or $x^4+y^4 = 1-2-1=-2$

which is not possible.

Hence $\vec{x}_{1,2} = \pm\sqrt{\sqrt{2}-1}\,(1,-1)$

are the only solutions.

Note that the curve is symmetric under the change $x \to -x$, $y \to -y$. Thus, one tangent line must touch the curve in a point with coordinates $(a,-a)$. Thus

$$2a^4+4a^2=2 \quad \text{or} \quad z^2+2z=1 \quad \text{with } z=a^2$$

which leads to the solution obtained before

Note that our previous calculation shows that there are no others.

To find the tangent line parallel to the $x=-y$ axis we must have that

$$x^3-y = y^3-x \quad \text{or} \quad (x^3-y^3)+(x-y)$$
$$= (x-y)\left[x^2+xy+y^2+1\right]=0$$

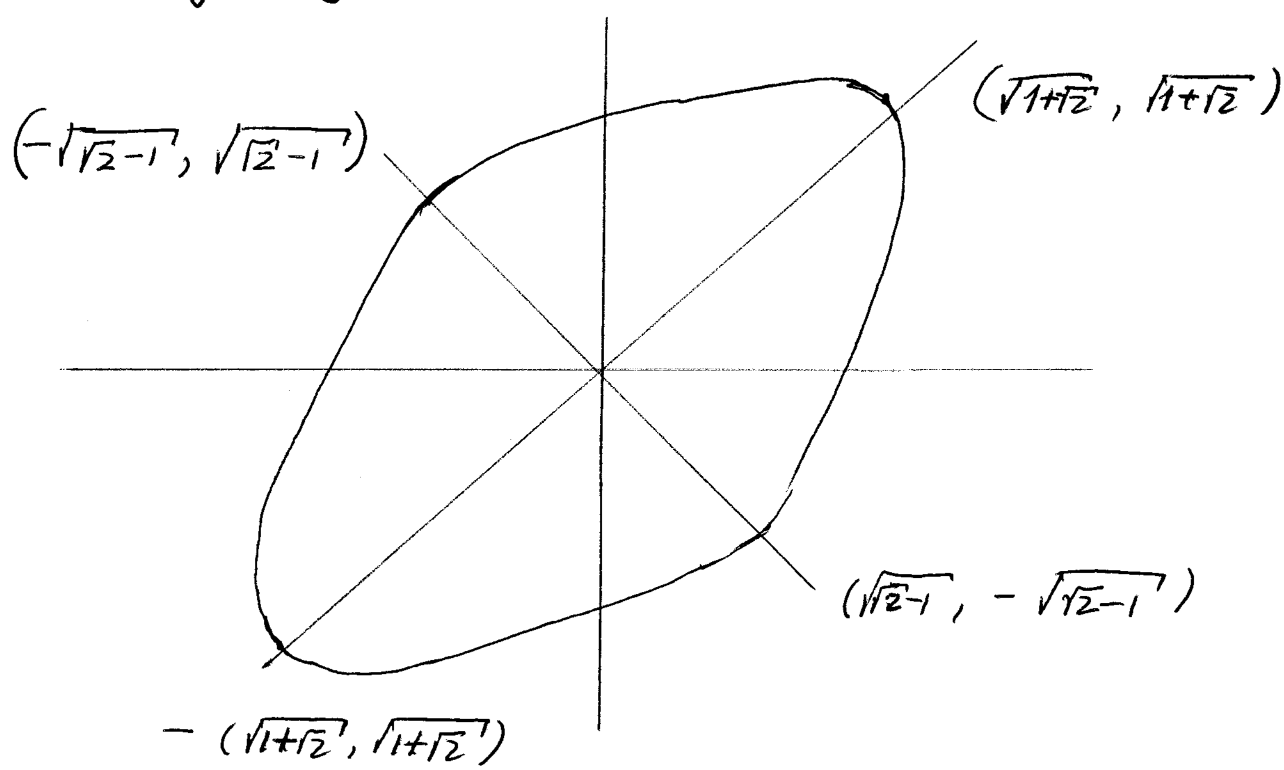
$x=y$ together with $x^4+y^4-4xy=2$ leads to $z^2-2z^2=1$ with $z=x^2$ from which we obtain

$$x=y=\pm\sqrt{1+\sqrt{2}}.$$

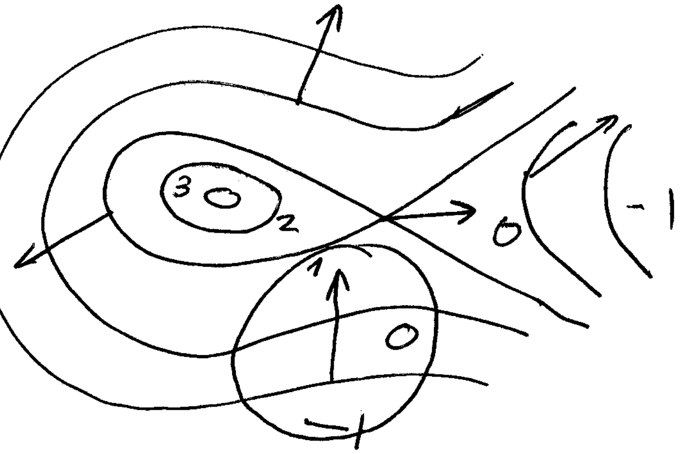
Note that $x^2+y^2+xy+1=0$ has no solution since $x^2+y^2 \geqslant 2|x||y|$ and hence

$$x^2+y^2+xy \geqslant 0.$$

f)



2) a)



The circled vector is the only one compatible with being a gradient.

b) $$\nabla f(\vec{x}) = \frac{\left((1-3x^2+y^2)y,\ (1-3y^2+x^2)x\right)}{(1+x^2+y^2)^3}$$

Crit. points:

i) $x=y=0$ $\quad \begin{bmatrix}0\\0\end{bmatrix}$ is a crit. point.

ii) $x \neq 0$, $y=0$ $\Rightarrow$ $1-3y^2+x^2=0$ or $1+x^2=0$ which is impossible.

Similarly $y \neq 0$ and $x=0$ is also impossible

iii) $1-3x^2+y^2=0$, $1-3y^2+x^2=0$ $\Rightarrow$ $x^2=1/2$, $y^2=1/2$

Crit. points: $\begin{bmatrix}0\\0\end{bmatrix}, \pm\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \pm\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

The Hessian at the crit. points:

At $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ At $\pm\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}$

At: $\pm\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

The eigenvalues are (in the same order)

± 1, $-4, -2$ $4, 2$

saddle local max. local min