Solutions to Practice Test 1 B

1.

a) $$\nabla f = \begin{bmatrix} 4x^3 + yz \\ 4y^3 + xz \\ 4z^3 + xy \end{bmatrix}$$

$$\nabla f(1,2,1) = \begin{bmatrix} 6 \\ 5 \\ 6 \end{bmatrix} \qquad f(1,2,1) =$$

$$1 + 16 + 1 + 2 = 20$$

b) $h(x,y,z) = 20 + 6(x-1) + 5(y-2) + 6(z-1)$

best linear approximation.

c) Choose variables (x, y, z, ω), then the equation for the tangent plane is

$$6(x-1) + 5(y-2) + 6(z-1) - (\omega - 20) = 0$$

Note the ~~most~~ vector normal to the plane is

$$\vec{N} = \begin{bmatrix} 6 \\ 5 \\ 6 \\ -1 \end{bmatrix}$$

d) $\vec{a}$ is tangent to the surface if $\vec{a} \cdot \vec{N} = 0$

Thus

$$\vec{a}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix}$$

are examples of such vectors.

2. a) Have to solve

$$x^3 - 2xy + 1 = 0 \quad \text{and} \quad x^3 - y^3 = (x-y)(x^2 + xy + y^2) = 0$$

i) $x = y$ and $x^3 - 2x^2 + 1 = 0$

Guess: $\underline{\underline{x = 1}}$ is a solution.

Long division $x^3 - 2x^2 + 1 : x - 1 = x^2 - x - 1$

$$\begin{array}{l} x^3 - 2x^2 + 1 \\ \underline{x^3 - x^2} \\ \quad -x^2 + 1 \\ \quad \underline{-x^2 + x} \\ \quad\quad -x + 1 \end{array}$$

$$x^2 - x - 1 = \left(x - \tfrac{1}{2}\right)^2 - \frac{5}{4} = 0$$

$$x = \frac{1+\sqrt{5}}{2}, \quad x = \frac{1-\sqrt{5}}{2}$$

Points of intersection:

$$\underset{(\mathrm{I})}{\underline{\underline{(1,1)}}} \quad \underset{(\mathrm{II})}{\underline{\underline{\left(\frac{1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right)}}}, \quad \underset{(\mathrm{III})}{\underline{\underline{\left(\frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)}}}.$$

ii) We have to check:

$$x^3 - 2xy + 1 = 0 \quad \text{and} \quad x^2 + xy + y^2 = 0$$

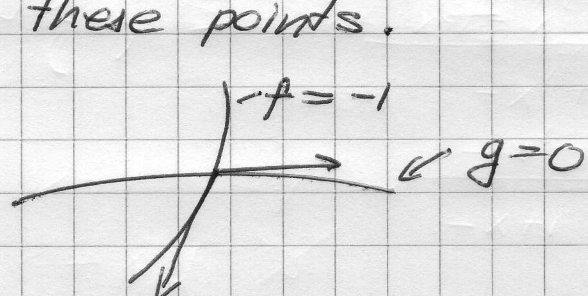
But

$$x^2 + xy + y^2 = \left(x + \tfrac{1}{2}y\right)^2 + \tfrac{3}{4}y^2 = 0$$

$x = y = 0$ is the only solution, which, however does not satisfy $x^3 - 2xy + 1 = 0$

Thus, the above three points are the only intersection points.

b) Have to find the tangent lines at these points.



Tangent vectors are perp. to the gradient.

$$\nabla f = \begin{bmatrix} 3x^2 - 2y \\ -2x \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 3x^2 \\ -3y^2 \end{bmatrix}$$

c) Tangent vectors

$$\vec{t}_f = \begin{bmatrix} 2x \\ 3x^2-2y \end{bmatrix} \qquad \vec{t}_g = \begin{bmatrix} 3y^2 \\ 3x^2 \end{bmatrix}$$

At (I):

$$\vec{t}_f = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{t}_g = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Tangent lines:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

At (II):

$$\vec{t}_f = \begin{bmatrix} 1+\sqrt{5} \\ \frac{1+\sqrt{5}}{2}+3 \end{bmatrix} \qquad \vec{t}_g = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (x = y\,!?)$$

$$\frac{1+\sqrt{5}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1+\sqrt{5} \\ \frac{7+\sqrt{5}}{2} \end{bmatrix} \qquad \frac{1+\sqrt{5}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

At (III):
$$\vec{t}_f = \begin{bmatrix} 1-\sqrt{5} \\ \frac{7-\sqrt{5}}{2} \end{bmatrix} \qquad \vec{t}_g = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1-\sqrt{5}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1-\sqrt{5} \\ \frac{7-\sqrt{5}}{2} \end{bmatrix} \qquad \frac{1-\sqrt{5}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

d) Angles: θ angle between the tangent vectors

~~cos θ~~

$$\cos\theta = \frac{\vec{t}_f \cdot \vec{t}_g}{|\vec{t}_f|\,|\vec{t}_g|}$$

At (I): $\cos\theta = \dfrac{3}{\sqrt{1+4}\,\sqrt{2}} = \dfrac{3}{\sqrt{5}\,\sqrt{2}}$

At (II): $\cos\theta = \dfrac{3\left(\frac{3+\sqrt{5}}{2}\right)}{\sqrt{39+\sqrt{5}}}$

At (III): $\cos\theta = \dfrac{3\left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{39-\sqrt{5}}}$.

3. $\nabla f = \begin{bmatrix} 2x \\ -2y \\ 0 \end{bmatrix} \quad \nabla g = 4\begin{bmatrix} x^3 \\ y^3 \\ z^3 \end{bmatrix}$

$\nabla f(1,0,1) = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \quad \nabla g(1,0,1) = 4\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

The vector tangent to the surface $f=1$ and $g=2$ at $(1,0,1)$ must be perp to the two gradients.

$\vec{t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ does the job.

Tangent line:

$$\underline{\underline{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}}$$

Extra credit:

The distance line must be perpendicular to $x^2 - y^2 = 4$ at some point $\vec{x}_0$ and to $y - 2x = 0$ at some point $\vec{x}_1$.

The direction of this line at $\vec{x}_0 = (u, v)$ is $\begin{bmatrix} 2u \\ -2v \end{bmatrix}$ this is the gradient!

and at $\vec{x}_1 = (\alpha, \beta)$ $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Thus

$$\begin{bmatrix} 2u \\ -2v \end{bmatrix} = \lambda \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{aligned} 2u &= -2\lambda \\ -2v &= \lambda \end{aligned}$$

or $\underline{2u = 4v}$ or $\underline{\underline{u = 2v}}$

Since $u^2 - v^2 = 4$ we have

$$4v^2 - v^2 = 4 \quad \text{or} \quad v^2 = \frac{4}{3}$$

$$v_1 = \frac{2}{\sqrt{3}}, \quad v_2 = -\frac{2}{\sqrt{3}},$$

Thus we have two candidates for $\vec{x}_0$:

$$\left(\frac{4}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) \text{ and } \left(-\frac{4}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$$

The normal lines are

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ and } -\frac{1}{\sqrt{3}}\begin{bmatrix} 4 \\ 2 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

point on $x^2 - y^2 = 4$ | point on $x^2 - y^2 = 4$

We have to find the points on $y = 2x$.

$$\frac{1}{\sqrt{3}}2 + s = 2\left(\frac{1}{\sqrt{3}}4 - 2s\right) \Rightarrow s = \frac{2\sqrt{3}}{5}.$$

and the point on $y = 2x$ is

$$\frac{8}{5\sqrt{3}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

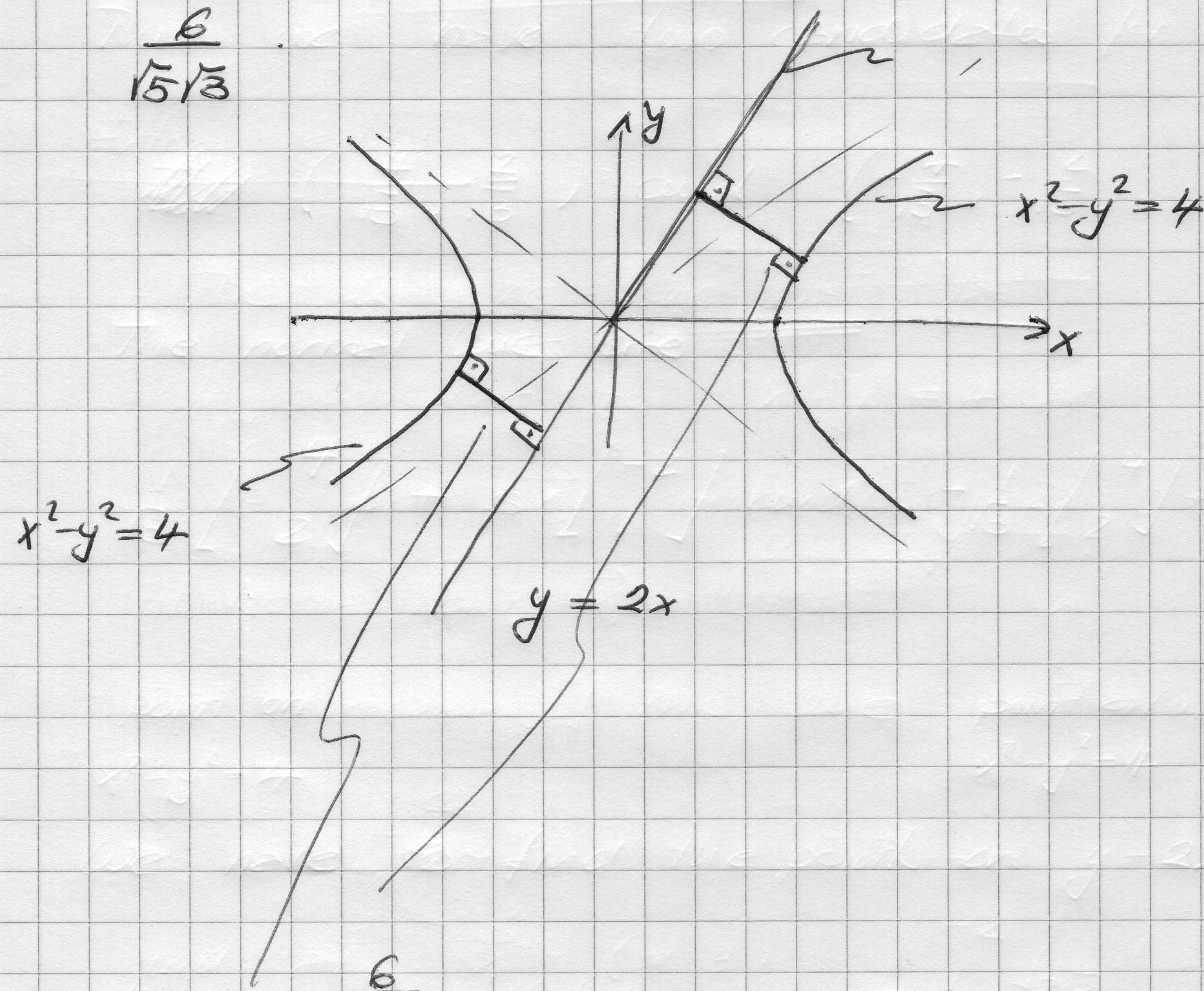
The vector connecting the two points $\frac{6}{5\sqrt{3}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

and the distance is

$$\frac{6}{5\sqrt{3}}\sqrt{5} = \frac{6}{\sqrt{5}\sqrt{3}}$$

The distance at the other points is also

$\frac{6}{\sqrt{5}\sqrt{3}}$.



$$\frac{6}{\sqrt{5}\sqrt{3}}$$