Consider the function $f(x, y) = xy + x^3$.

I: (3 points) Find the equation of the plane that is tangent to the graph of $f$ at the point $(1, 2)$.

\[
\nabla f(x, y) = \begin{bmatrix} y + 3x^2 \\ x \end{bmatrix} \quad \nabla f(1, 2) = \begin{bmatrix} 5 \\ 1 \end{bmatrix}
\]

\[
z = 3 + 5(x - 1) + (y - 2)
\]

II: (3 points) Find the line that is tangent to the level curve of the function $f$ at the point $(1, 2)$. Give the line in parametrized form.

Direction vector is

\[
\begin{bmatrix} -1 \\ 5 \end{bmatrix}
\]

which is perpendicular to $\nabla f(1, 2)$. The line is therefore given by

\[
\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \end{bmatrix}
\]

III: (2 points) Find the rate of change of the function $f(x, y)$ at the point $(1, 2)$ in the direction $(2, 2)$.

\[
\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 12.
\]

IV: (2 points) Find all the critical points of the function $f(x, y)$.

\[
\nabla f(x, y) = \begin{bmatrix} y + 3x^2 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
y + 3x^2 = 0, \quad x = 0
\]

$(0, 0)$ is the only solution.

Extra credit: (3 points) Find the curvature of the function $f$, i.e., the second derivative at $t = 0$ of the function $g(t) = f(x_0 + tv)$ where $x_0 = (1, 2)$ and $v = (2, 1)$.

The Hessian is

\[
H_f(x, y) = \begin{bmatrix} 6x & 1 \\ 1 & 0 \end{bmatrix}
\]

\[
H_f(1, 2) = \begin{bmatrix} 6 & 1 \\ 1 & 0 \end{bmatrix}
\]

and

\[
g''(0) = 28.
\]