Test 1 for Calculus III for CS Majors, Math 2506 J1-J2, September 25, 2007

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: Consider the function

$$f(x,y) = x^3 + y^3 - 3xy \ .$$

a) (5 points) Calculate the gradient at the point $(1,-1)$.

Answer:

$$\nabla f(x,y) = 3\begin{bmatrix} x^2 - y \\ y^2 - x \end{bmatrix}$$

and

$$\nabla f(1,-1) = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \ .$$

b) (5 points) Find the line (in parametrized form) that is tangent to the curve $f(x,y) = f(1,-1)$ at the point $(1,-1)$.

Answer: The tangent vector is given by

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

and hence the tangent line is given by

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

c) (5 points) Find the best linear approximation of the function $f(x,y)$ at the point $(1,-1)$.

Answer:

$$h(x,y) = 3 + 6(x-1) \ .$$

d) (10 points) Find the points on the curve $f(x,y) = f(1,-1)$ where the tangent line is horizontal.

Answer: Tangent line horizontal means that the second component of the tangent vector vanishes, which in turn means that the first component of the gradient vanishes. Thus

$$x^2 = y \ , \ x^3 + y^3 - 3xy = 3$$

which means that

$$x^6 - 2x^3 = 3$$

Setting $s = x^3$ we have $s^2 - 2s - 3 = 0$ or $(s-1)^2 - 4 = 0$ or $s_1 = 3$ $s_2 = -1$. $x = 3^{1/3}, -1$ and we have the points

$$(3^{1/3}, 3^{2/3}) \ , \ (-1, 1) \ .$$

Problem 2: a) (10 points) Calculate the critical points of the function

$$f(x, y) = x^3 + yx^2 - \frac{1}{2}x^2 - y \ .$$

Answer:

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 + 2xy - x \\ x^2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which yields the critical points

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} , \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

b) (10 points) Calculate the Hessian at these critical points.

Answer:

$$H_f(x, y) = \begin{bmatrix} 6x + 2y - 1 & 2x \\ 2x & 0 \end{bmatrix}$$

which yields

$$\begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -3 & -2 \\ -2 & 0 \end{bmatrix}$$

c) (10 points) What are the type of these critical points, are they a max a min or a saddle?

Answer: The determinant is negative in both cases and hence the critical points are saddles.

Problem 3: A function $g(x, y)$ has $(0, 0)$ as a critical point and the Hessian at this point is given by

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} .$$

a) (5 points)Write the quadratic approximation $q(x, y)$ for the function $g(x, y)$ in the vicinity of this critical point.

Answer:

$$x^2 + 3xy + y^2$$

b) (10 points) Find the eigenvalues and the eigenvectors of the Hessian.

Eigenvalue 5 with eigenvector

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and eigenvalue -1 with eigenvector

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

c) (10 points) Draw in a qualitative fashion a few of the level curves of $q(x, y)$.

Problem 4: (10 points) a) Set up Newton's scheme for solving the equation $x^2 - y = 1$ and $xy = 1$.

Answer:

$$\mathbf{x_1} = \mathbf{x_0} - J_F^{-1}(\mathbf{x_0})F(\mathbf{x_0})$$

where

$$F(\mathbf{x}= \begin{bmatrix} x^2 - y - 1 \\ xy - 1 \end{bmatrix}$$

and

$$J_F(\mathbf{x}) = \begin{bmatrix} 2x & -1 \\ y & x \end{bmatrix}$$

b) (10 points) Use as an initial guess the point $\mathbf{x_0} = (1, 1)$ and calculate the next approximation $\mathbf{x_1}$. Check whether this leads to an improvement.

Answer: We have to calculate

$$\mathbf{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$|F(\mathbf{x_1})| = \sqrt{2}/9$$

which is an improvement.

Extra Credit: (15 points) Given the function $f(x, y) = x^3 - 3xy^2$ and $g(x, y) = 3x^2y - y^3$. What can you say about the angles between lines tangent to the level curves of f resp. g at any point (x, y)?

Answer:

$$\nabla f = \begin{bmatrix} 3(x^2 - y^2) \\ -6xy \end{bmatrix} \;,\; \nabla g = \begin{bmatrix} 6xy \\ 3(x^2 - y^2) \end{bmatrix} \;.$$

$$\nabla f \cdot \nabla g = 0$$

The level curves intersect in a right angle.