Test 2 for Calculus III for CS Majors, Math 2506 J1-J2, October 23, 2007

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: (10 points) a) Diagonalize the matrix $A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$, i.e., find an invertible matrix S and a diagonal matrix D so that $A = SDS^{-1}$.

$$A = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

b) (10 points) Find a Schur factorization for the same matrix A, i.e., find an orthogonal matrix Q and an upper triangular matrix T so that $A = QTQ^T$.

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Problem 2: Consider the matrix

$$A = \begin{bmatrix} 6 & 0 & 4 \\ 0 & 3 & 1 \\ 4 & 1 & 0 \end{bmatrix}$$

a) (5 points) Compute $\mathrm{Off}(A)$.

34

b) (10 points) Calculate the Givens matrix G for the first step in the Jacobi iteration for diagonalizing A by picking the 2×2 submatrix with the largest off diagonal elements. (You do not have to calculate $G^T AG$.)

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 0 & -1 \\ 0 & \sqrt{5} & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

c) (10 points) Calculate $\mathrm{Off}(G^T AG)$.

2

Problem 3: a) (15 points) Calculate the singular value decomposition of the matrix

$$A = \frac{1}{3\sqrt{2}} \begin{bmatrix} 6 & 2 \\ 3 & 5 \\ 0 & 4 \end{bmatrix}$$

$$A^T A = \frac{1}{2} \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = \frac{1}{3} \begin{bmatrix} 2 & -2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$$

b) (10 points) Find the best rank one approximation.

$$\frac{1}{3\sqrt{2}} \begin{bmatrix} 4 & 4 \\ 4 & 4 \\ 2 & 2 \end{bmatrix}$$

Problem 4: Assume that

$$V = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, U = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

form the singular decomposition of a matrix B.

a) (10 points) Calculate the generalized inverse B^{+}

$$A^{+} = UD^{-1}V^{T} = \frac{1}{18\sqrt{5}}\begin{bmatrix} -10 & -16 & -2 \\ 10 & -2 & 11 \end{bmatrix}$$

b) (5 points) Find the least square solution of the problem $Bx = b$ where

$$b = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Problem 5: (15 points) Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3$ in the region consisting of all points (x, y) that satisfy $x^2 + y^2 - xy \leq 1$. Find all the points where the maximum is taken and all the points where the minimum is taken. (Hint for the calculation: The identity $x^3 - y^3 = (x - y)(x^2 + y^2 + xy)$ might be useful.)

Critical point:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Lagrange multiplier equation:

$$\begin{bmatrix} x^2 \\ y^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x - y \\ 2y - x \end{bmatrix}$$

By cross-multiplying we get

$$x^3 - y^3 = 2xy(x - y)$$

or

$$(x - y)[x^2 + y^2 + xy] = 2xy(x - y) \ .$$

Either $x = y$ in which case we get from the equation

$$x^2 + y^2 - xy = 1$$

the point $\pm(1, 1)$. Or $x \neq y$ in which case we have that $0 = x^2 + y^2 - xy$ which is not possible. Max is at $(1, 1)$ with value 2 and min is at $-(1, 1)$ with value -2.

Extra Credit: (10 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 0.1 & -0.1 \\ 0.2 & 1 & 0.1 \\ -0.2 & 0.1 & 2 \end{bmatrix}$$

Find upper and lower bounds on the eigenvalues using the 'Small Gershgorin Disk Theorem'.

Since $\delta(A) = 1, r(A) = 0.3$,

$$\frac{2r(A)^2}{\delta(A)} = 0.18 \ .$$

The intervals for the eigenvalues are $[-0.18, 0.18], [0.82, 1.18], [1.82, 2.18]$.