Test 3 for Calculus III for CS Majors, Math 2506 J1-J2, November 20, 2007

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Euler's formula

$$Q(\theta, \mathbf{u}) = \cos\theta I + (1-\cos\theta)\mathbf{u}\mathbf{u}^T + \sin\theta B_{\mathbf{u}}$$

where

$$B_{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Problem 1: Consider the matrix $A = \begin{bmatrix} 3 & 3 \\ 4 & -1 \end{bmatrix}$

a) (5 points) Find the QR factorization of A.

$$A = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$

b) (10 points) Find the Schur factorization of A.

$$U = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$$

and

$$A = U^T \begin{bmatrix} 5 & -1 \\ 0 & -3 \end{bmatrix} U^T$$

c) (10 points) Calculate e^{At}. (Do not waste time simplifying the answer)

$$e^{At} = \begin{bmatrix} \frac{3}{4}e^{5t} + \frac{1}{4}e^{-3t} & \frac{3}{8}(e^{5t} - e^{-3t}) \\ \frac{1}{2}(e^{5t} - e^{-3t}) & \frac{1}{4}e^{5t} + \frac{3}{4}e^{-3t} \end{bmatrix}$$

Problem 2: Consider the system of differential equations $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ where

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} y^2 - 5x + 6 \\ x - y \end{bmatrix} .$$

a) (5 points) Find the equilibrium points (also known as critical points) of this system.

$$(2, 2) \ , \ (3, 3) \ .$$

b) (5 points) Linearize the system at each of these critical points.

$$\begin{bmatrix} -5 & 4 \\ 1 & -1 \end{bmatrix} , \begin{bmatrix} -5 & 6 \\ 1 & -1 \end{bmatrix}$$

c) (10 points) Determine the type and stability of each of these critical points for the linear system

The first is stable since the determinant is positive and the trace is negative. The second is a saddle since the determinant is negative.

d) (5 points) Determine the type and stability of each of these critical points for the non-linear system.

Problem 3: The following matrix is a rotation

$$Q = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

a) (5 points) Find the angle of rotation.

$$\cos\theta = \frac{1}{3}$$

b) (10 points) Find the axis of rotation

$$\sin\theta = \frac{2\sqrt{2}}{3}$$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

d) (10 points) Find the matrix that rotates about the axis $\mathbf{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ by an angle $2\pi/3$. (Simplify the answer as much as possible.)

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 4: Consider the curve

$$\mathbf{x}(t) = \begin{bmatrix} \sqrt{2}t \\ e^t \\ e^{-t} \end{bmatrix} , 0 \leq t \leq 2 \ .$$

a) (10 points) Find the length of this curve.

$$\int_0^2 (e^t + e^{-t})dt = e^2 - e^{-2} \ .$$

b) (5 points) Find the unit tangent vector at each point of this curve

$$\mathbf{T}(t) = \frac{1}{e^t + e^{-t}} \begin{bmatrix} \sqrt{2} \\ e^t \\ -e^{-t} \end{bmatrix}$$

c) (10 points) Find the curvature at every point of this curve.

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{x}'(t)|} = \frac{\sqrt{2}}{(e^t + e^{-t})^2} \ .$$

Extra Credit: (10 points) A rotation of $\pi/2$ about the x-axis is followed by a rotation of $\pi/2$ about the z-axis. Find the angle and axis of the resulting rotation.

The rotation is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Angle of rotation is $2\pi/3$ axis of rotation is

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$