Analysis Homework

Soniya Khan, Russel Krenek, Brendy Gallien, Matias Perez

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1)

$\int_0^1 xdx = \frac{1}{2}$ proof:

Let the partition $P = \left\{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$. Therefore we have that $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$

Now since $f(x)$ is an increasing function on the interval $[0, 1]$ we have that $U_f = \sum_{i=1}^{n} l.u.b\,[x_i, x_{i-1}]\, f(x)\frac{1}{n} = \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i = \frac{1}{n^2}\frac{(n+1)n}{2}$ and

$L_f = \sum_{i=1}^{n} g.l.b.\,[x_i, x_{i-1}]\, f(x)\frac{1}{n} = \sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} = \frac{1}{n^2}\sum_{i=1}^{n} i - 1 = \frac{1}{n^2}\frac{(n-1)n}{2}$

Therefore by taking the limit as n goes to infinity we get

$I_U = \lim_{n\to\infty} U_f = \lim_{n\to\infty} \frac{1}{n^2}\frac{(n+1)n}{2} = \frac{1}{2}$

and similarly we have

$I_L = \lim_{n\to\infty} L_f = \lim_{n\to\infty} \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{1}{2}$

Thus we have that $\int_0^1 xdx = \frac{1}{2}$

2)

We want to show that $\int_0^1 f(x)dx = 0$ if $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{Z}$ and $f(x) = 0$ for all other x.

Proof:

It is tempting to say that this statement is false. We may look at it in the following way and reach this fallacy. Let us consider the patrition $P = \left\{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$ so that $\Delta x_i = x_i - x_{i-1} = \frac{1}{n}$. And now we use the Darboux definition for an integrable function. We let

$U_{f,P} = \sum_{i=1}^{n} \sup_{[x_i, x_{i-1}]} f(x)\frac{1}{n}$

and we let

$L_{f,P} = \sum_{i=1}^{n} \inf_{[x_i, x_{i-1}]} f(x)\frac{1}{n}$

now we have

$U_{f,P} = \sum_{i=1}^{n} \frac{1}{n} = \frac{1}{n}n = 1$

and

$L_{f,P} = 0$

since $\sup_{[x_i,x_{i-1}]} f(x) = 1$ and $\inf_{[x_i,x_{i-1}]} f(x) = 0$. Now we take
$I_U = \inf U_{f,P} = 1$
and
$I_L = \sup L_{f,P} = 0$

This clearly tells us that the statement made in problem 2 is wrong, since $I_L \neq I_U$ but this logic is incorrect. We assumed something that was not true and that is that we can find $f(x) = 1$ for every partition in $[0, 1]$. Lets consider that we take $\left[\frac{4}{6}, \frac{5}{6}\right]$, then $x \neq \frac{1}{n}$ in this interval. But we summed $\sum_{i=1}^{n} \frac{1}{n} = \frac{1}{n} n = 1$ considerring that $\sup_{[x_i,x_{i-1}]} f(x) = 1$, n times. So now we must look for a better partition, one that will tell us exactly how many time $\sup_{[x_i,x_{i-1}]} f(x) = 1$. If we want $\sup_{[x_i,x_{i-1}]} f(x) = 1$, $n+1$ times we must have that $\Delta x_i = \frac{1}{n} - \frac{1}{n+1}$ since if $k < n$ then $\frac{1}{k} - \frac{1}{k+1} > \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. So now we take the partition $P_2 = \left\{\frac{1}{n(n+1)}, \frac{2}{n(n+1)}, ..., \frac{n^2+n-1}{n(n+1)}, \frac{n(n+1)}{n(n+1)}, \right\}$and we let $\Delta x_i = \frac{1}{n(n+1)}$. So now we have

$$U_{f,P_2} = \sum_{i=1}^{n(n+1)} \sup_{[x_i,x_{i-1}]} f(x) \frac{1}{(n+1)n} = \sum_{i=1}^{n+1} \frac{1}{(n+1)n} = \frac{1}{(n+1)n}(n+1) = \frac{1}{n}$$

since $\sup_{[x_i,x_{i-1}]} f(x) = 1$, $n+1$ times and $f(x) = 0$ in every other partition.

and

$$L_{f,P_2} = \sum_{i=1}^{n(n+1)} \inf_{[x_i,x_{i-1}]} f(x) \frac{1}{n} = 0$$

So we have

$I_U = \inf U_{f,P_2} = \inf \frac{1}{n} = 0$ and $I_L = \sup L_{f,P_2} = 0$ and we are done.

3)

Does $\int_0^1 f(x)dx$ exist for

$$f(x) = \begin{cases} \frac{1}{q} \text{ if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ and } GCD(p, q) = 1 \\ 0 \text{ if } x \notin \mathbb{Q} \end{cases}$$

We will claim that $\int_0^1 f(x)dx$ does exisit

In this problem we will use the Riemann definition of integrable functions. This means that for any $\epsilon > 0$ there exist a $\delta > 0$ such that $|S - A| < \epsilon$ for $|\Delta x| < \delta$. Where S is a Riemann sum of $f(x)$ and A is the Riemann integral of $f(x)$.

We must show that for any Riemann integral the definition above holds. It is obvios from the last proble that the Riemann sum goes to zero for the $\inf_{[x_{i+1},x_i]} f(x)$ since $\inf_{[x_{i+1},x_i]} f(x) = 0$. Now we must show that any Riemann sum of $f(x)$ must also go to 0. for this we will take the Riemann sum on the $\sup_{[x_{i+1},x_i]} f(x)$ and show that it is smaller the ϵ. First we must note that when the function takes the value of $\frac{1}{n}$ there are finitly many $x \in [0, 1]$ such that $f(x) > \frac{1}{n}$. Therefore there are finitly many points $x \in [0, 1]$ such that $f(x) > \frac{\epsilon}{2}$ and we can name these points x_i for $i \in \{1, 2, 3, ..., N-1, N\}$. Now lets consider the following partiotion, let P be a partition centered at x_i $\left[x_i - \frac{\epsilon}{4N}, x_i + \frac{\epsilon}{4N}\right]$ and let $|\Delta x| = \frac{\epsilon}{2N}$ where N is the finite number of points greater then $\frac{\epsilon}{2}$. Therefore now we have that

$$|S| < \frac{\epsilon}{2} + \sum_{i=0}^{N} \frac{\epsilon}{2N} = \frac{\epsilon}{2} + \frac{\epsilon}{2N}N = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

7)

We want to show that if $f : [0,1] \to \mathbb{R}$ is bounded and not continuous at a finite number of points then for $\epsilon > 0$ there exist a $\delta < 0$ such that $|\Delta x_i| < \delta$ implies that $|S_f - A| < \epsilon$ where P is a partition of $[a,b]$.

Now let $t_1, t_2, ..., t_{N-1}, t_N$ be the points in $[a,b]$ such that $f(t_i)$ is not continuous for all $i \in \{1,2,3,...,N-1,N\}$

Now let f_1 and f_2 be defined as

$$f_1(x) = \begin{cases} l.u.b.\{f(x)|x \in (x_i, x_{i+1})\} \text{ for } i \in \{1,2,3,...,n-1,n\} \\ f(x) \text{ if } x = x_i \end{cases}$$

$$f_2(x) = \begin{cases} g.l.b.\{f(x)|x \in (x_i, x_{i+1})\} \text{ for } i \in \{1,2,3,...,n-1,n\} \\ f(x) \text{ if } x = x_i \end{cases}$$

where $t_i = x$ for $x \in \{x_1, x_2, ..., x_{n-1}, x_n\}$

Now since $f(x)$ is bounded we have $f_2(x) \leq f(x) \leq f_1(x)$. Now we have that since $f(x)$ is continuous in (x_i, x_{i+1}) we have that for any $x', x" \in (x_i, x_{i+1})$ that $|x" - x'| < |x_i - x_{i+1}| < \delta$

so that

$$|f(x") - f(x')| < |\sup_{(x_i,x_{i+1})} f(x) - \inf_{(x_i,x_{i+1})} f(x)| < \frac{\epsilon}{b-a}.$$

which implies that $|f_1(x) - f_2(x)| < \dfrac{\epsilon}{b-a}$

Now we get that

$$\int_a^b (f_1(x) - f_2(x))dx < \max\{f_1 - f_2\} \dot{(}b-a) < \frac{\epsilon}{b-a}(b-a) = \epsilon$$

8)

Show that if $f(x) : [a,b] \to \mathbb{R}$ is stricly increasing (or decreasing) we have that $f(x)$ is integrable.

Proof:

Now since $f(x)$ is increasing on $[a,b]$ and $[a,b]$ is bounded then $f(b) \geq f(x)$ and $f(a) \leq f(x)$ for all $x \in [a,b]$. Noww lets consider the partition $P = \left\{\frac{a}{n}, \frac{a+1}{n}, \frac{a+2}{n}, ..., \frac{a+n-1}{n}, b\right\}$ so that we have that $\delta x_i = \frac{1}{n}$ now we take

$$U_f = \sum_{i=1}^{n} \sup_{[x_i,x_{i-1}]} f(x)\frac{1}{n} = \sum_{i=1}^{n} x_i \frac{1}{n}$$

and we let

$$L_f = \sum_{i=1}^{n} \sup_{[x_i,x_{i-1}]} f(x)\frac{1}{n} = \sum_{i=1}^{n} x_{i-1} \frac{1}{n}$$

Now since $f(x)$ is increasing and $[a,b]$ is boundedthen we have that there exist a M such that $|f(b) - f(a)| \geq M$. There fore we have that

$|U_f - L_f| < \dfrac{f(b) - f(a)}{n} \leq \dfrac{2M}{n}$. Now take $n \to \infty$ and we are done.

9)

We want to show that if the function $f : [a,b] \to \mathbb{R}$ is integrable on the interval $[a,b]$ then so is $|f|$, and $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$.

Proof:

So now we look at

$U_{|f|,P} - L_{|f|,P} = \sum_{i=1}^{n} \sup_{[x_i,x_{i-1}]} f(x)\Delta x_i - \sum_{i=1}^{n} \inf_{[x_i,x_{i-1}]} f(x)\Delta x_i =$

$\sum_{i=1}^{n} (\sup_{[x_i,x_{i-1}]} f(x) - \inf_{[x_i,x_{i-1}]} f(x))\Delta x_i$

Now from definition we have that for $z, y \in A$ we have that

$\sup_A |f(x)| \leq |f(y)| + \epsilon$

$\inf_A |f(x)| \geq |f(z)| - \epsilon$

so now we have

$\sup_A |f(x)| - \inf_A |f(x)| \leq |f(y)| + \epsilon - |f(z)| + \epsilon$

$\leq |f(y) - f(z)| + 2\epsilon \leq \sup_A f(x) - \inf_A f(x) + 2\epsilon$

thus

$\sup_A |f(x)| - \inf_A |f(x)| \leq \sup_A f(x) - \inf_A f(x)$

and

$f(y) - f(z) \leq \sup_A f(x) - \inf_A f(x)$

$f(z) - f(y) \leq \sup_A f(x) - \inf_A f(x)$

which implies

$|f(y) - f(z)| \leq \sup_A f(x) - \inf_A f(x)$

Now we have that $U_{|f|,P} - L_{|f|,P} \leq U_{f,P} - L_{f,P}$

therefore if $U_{f,P} - L_{f,P}$ converges then so does $U_{|f|,P} - L_{|f|,P}$ so therefore $|f(x)|$ is also integrable.

Now we have $-\sup f(x) \leq \sup f(x)$ since we have that

$-|f(x)| \leq f(x) \leq |f(x)|$. Thus we have

$U_{-|f|,P} \leq U_{f,P}$, so $\int_a^b -|f(x)|dx = I_{-|f|,P} \leq I_{f,P} = \int_a^b f(x)dx$

Now we also have that $\sup f(x) \leq \sup |f(x)|$ thus we have $U_{f,P} \leq U_{-|f|,P}$

So $\int_a^b f(x)dx = I_{f,P} \leq I_{|f(x)|} = \int_a^b |f(x)|dx$

So therefore we have

$\int_a^b -|f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$

and therefore we have $|\int_a^b f(x)dx| \leq \int_a^b |f(x)|dx$ and we are done.

10)

We want to show that is $f(x)$ is integrable in $[a, b]$ then $f^2(x)$ is integrable in $[a, b]$. We will also show, with this result and the identitiy $(a+b)^2 = a^2+2ab+b^2$, that if $f(x)$ and $g(x)$ is integrable on $[a, b]$ then $g(x)f(x)$ is also integrable on $[a, b]$.

proof 1:

Now we will divide this proof into two cases, case 1 when $f(x) \geq 0$ and case 2 when $f(x) < 0$.

case 1:

Since $f(x)$ is integrable, we have that there exist $f_1(x)$ and $f_2(x)$ step function such that

$f_2(x) \leq f(x) \leq f_1(x)$ and therefore

$f_2^2(x) \leq f(x) \leq f_1^2(x)$

And since $f(x)$ is an integrable function then

$U_{f,P} - L_{f,P} < \frac{\sqrt{\epsilon}}{b-a}$ so we can pick $f_1(x) - f_2(x) < \frac{\sqrt{\epsilon}}{b-a}$.

Therefore we have that

$(f_1(x) - f_2(x))^2 < \frac{\epsilon}{b-a}$. So therefore we have that

$\int_a^b (f_1(x) - f_2(x))^2 dx =$

$\int_a^b (f_1^2(x) - 2f_1(x)f_2(x) + f_2^2(x))dx < \int_a^b (f_1^2(x) - 2f_2^2(x) + f_2^2(x))dx = \int_a^b (f_1^2(x) - f_2^2(x))dx < \frac{\epsilon}{b-a}(b-a) = \epsilon$

case 2:

Since $f(x) < 0$ we have that

$f_1^2(x) \leq f(x) \leq f_2^2(x)$

and now we pick our step function such that

$f_1(x) - f_2(x) < \frac{\sqrt{\epsilon}}{b-a}$

Therefore we have that

$(f_2(x) - f_1(x))^2 < \frac{\epsilon}{b-a}$

So therefore we have that

$\int_a^b (f_1(x) - f_2(x))^2 dx =$

$\int_a^b (f_2^2(x) - 2f_1(x)f_2(x) + f_1^2(x))dx < \int_a^b (f_2^2(x) - 2f_1^2(x) + f_1^2(x))dx = \int_a^b (f_2^2(x) - f_1^2(x))dx < \frac{\epsilon}{b-a}(b-a) = \epsilon$

Proof 2:

Now lets consider that $f(x)$ and $g(x)$ is integrable, then we have that $f(x) - g(x) = h(x)$ then $h(x)$ is integrable and using the result from the beggining of this problem we have that $h^2(x)$ is also integrable. So then

$\int_a^b h^2(x)dx = \int_a^b (f(x) - g(x))^2 dx = \int_a^b (f^2(x) - 2f(x)g(x) + g^2(x))dx$

and therefore

$\int_a^b g(x)f(x)dx = \frac{\int_a^b (f^2(x) + g^2(x))dx}{2}$

and since $f^2(x)$ and $g^2(x)$ we are done.

11)

Prove that if f is a continuous real value function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $f(x) > 0$ for some $x \in [a, b]$ then the $\int_a^b f(x)dx > 0$.

Proof:

Since f is a continuous value function on some integral f must be bounded and integrable. We choose some $x_0 \in [a, b]$ such that $\epsilon = f(x_0) > 0$. Since f is continuous, there is a $\delta > 0$ such that

$|f(x) - f(x_0)|, \epsilon$, whenever $|x_0 - x| < \delta$

therefore we have $-f(x_0) < 0$ will imply $0 < f(x) < 2\epsilon$ whenever $|x_0 - x| < \delta$

We know we can find an integer m such that $\frac{1}{m} < \delta$. If we make our partition P_n where $n \geq 2m$, then at least one partition will be completly contained in the interval $[x_0 - \delta, x_0 + \delta]$, which is an interval where $f(x) > 0$ everywhere.

Thus, $L_{f,P_n} = \sum_{i=1}^{n} \min_{[\frac{i-1}{n}, \frac{i}{n}]} \frac{f(x)}{n}$ has at least on term greater then 0 and no negative terms.

Therefore $L_{f,P_n} > 0$ for all $n \geq 2m$, emd

$I_L = \lim_{n \to \infty} L_{f,P_n} > 0$

Since f is integrable, $I_L = I_U = I$, so $I > 0$