A good source for this material is the book by Reed and Simon, Methods of Modern Mathematical Physics, Vol. I on Functional Analysis, which we follow.

1 The Diagonal Argument

1.1 DEFINITION (Subsequence). A subsequence of a given sequence is a function \( m : \mathbb{N} \to \mathbb{N} \) which is strictly increasing.

1.2 THEOREM. Consider a sequence of functions \( \{ f_n(x) \}_{N}^{\infty} \) defined on the positive integers that take values in the reals. Assume that this sequence is uniformly bounded, i.e., there is a positive constant such that

\[
|f_n(x)| \leq C
\]

for all \( n = 1, 2, \ldots \) and all \( x \in \mathbb{N} \). Then there exists a subsequence \( m(j) \) such that \( f_{m(j)} \) converges for all \( x \in \mathbb{N} \).

Proof. Since \( f_n(1) \) is a bounded sequence, there exists a subsequence \( f_{n_1(j)} \) of functions such that \( f_{n_1(j)}(1) \) converges as \( j \to \infty \). Now we pick a subsequence of \( n_1(j) \) which we call \( n_2(j) \) such that the sequence of functions \( f_{n_2(j)}(x) \) converges for \( x = 2 \). Proceeding in an inductive fashion we obtain a subsequence \( n_k(j) \) of the sequence \( n_{k-1}(j) \) such that the for the sequence of functions \( f_{n_k(j)}(x), f_{n_k(j)}(k) \) is convergent. Note, that this construction guarantees that \( f_{n_k(j)}(r) \) converges for all \( r \leq k \). Now we set

\[
m(j) = n_j(j),
\]

i.e., we pick the ‘diagonal sequence’. Note that \( f_{m(j)}(k) \) converges for every \( k \), since the sequence

\[
f_{m(k)}(k), f_{m(k+1)}(k), f_{m(k+2)}(k), f_{m(k+3)}(k) \ldots
\]

is a subsequence of the sequence \( f_{n_k(j)}(k) \), which converges. Hence \( f_{m(j)}(k) \) converges for all \( k = 1, 2, 3, \ldots \). For every \( k \), there are finitely many terms that are not part of the subsequence \( f_{n_k(j)}(k) \), namely

\[
f_{m(1)}(k), f_{m(2)}(k), f_{m(3)}(k) \ldots f_{m(k-1)}(k),
\]

but they are immaterial for the convergence of the sequence.
2 The \( \varepsilon/3 \) argument

2.1 THEOREM. The space \( C([0, 1]) \) consisting of continuous functions \( f : [0, 1] \to \mathbb{R} \) with metric

\[
D(f, g) = \max_{0 \leq x \leq 1} |f(x) - g(x)|
\]

is a complete metric space.

Proof. We have learned before that \( C([0, 1]) \) is a metric space. We have to worry about completeness. Let \( f_n(x) \in C([0, 1]) \) be a Cauchy Sequence. Thus, for every \( \varepsilon > 0 \) there exists \( N \) such that for all \( n, m > N \)

\[
\max_{0 \leq x \leq 1} |f_n(x) - f_m(x)| < \varepsilon/2 .
\]

In particular, for every fixed \( x \in [0, 1] \), \( f_n(x) \) is a Cauchy Sequence of real numbers and since the reals are complete, this sequence has a limit which we denote by \( f(x) \). Since for any \( m \)

\[
|f(x) - f_m(x)| = \lim_{n \to \infty} |f_n(x) - f_m(x)| \leq \text{l.u.b.}\{ |f_n(x) - f_m(x)| : n > N \} ,
\]

we have that for all \( m > N \)

\[
|f(x) - f_m(x)| \leq \varepsilon/2 < \varepsilon . \tag{2.1}
\]

Note that \( x \) is arbitrary and that \( \varepsilon \) is independent of \( x \), i.e., the convergence is uniform. Although we know from previous arguments that the limit must be continuous, let us prove this, because this uses a typical \( \varepsilon/3 \) argument. \( \varepsilon > 0 \). We have seen that there exists \( N \) so that for all \( n > N \) and all \( x \in [0, 1] \),

\[
|f(x) - f_n(x)| < \varepsilon/3
\]

Fix such a value for \( n \) and fix \( x \). Since \( f_n \) is continuous, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( y \in [0, 1] \) with \( |x - y| < \delta \) we have that

\[
|f_n(x) - f_n(y)| < \varepsilon/3 .
\]

Since we also have that

\[
|f_n(x) - f_n(y)| < \varepsilon/3 ,
\]

we may use the triangle inequality

\[
|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(x) - f_n(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon
\]
Thus, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( y \) is such that \( |x - y| < \delta \), then \( |f(x) - f(y)| < \varepsilon \). Thus, the limit \( f \) is continuous. Note, that from (2.1) we know that for any \( \varepsilon > 0 \) there exists \( N \) such that for all \( n > N \)

\[
|f(x) - f_n(x)| \leq \varepsilon/2
\]

and hence

\[
D(f, f_n) = \max_{0 \leq x \leq 1} |f(x) - f_n(x)| \leq \varepsilon/2 < \varepsilon ,
\]

and hence the sequence \( f_n \) converges to \( f \) in the metric \( D(f, g) \). \( \square \)

3 Equicontinuity and the Theorem of Arzela-Ascoli

We have seen various notions of continuity but they all were statements about a single function. In this section we shall talk about the continuity properties of a family of functions. In what follows we shall always consider two metric spaces \( E, E' \) and \( \mathcal{F} \) a family of continuous functions from \( E \) to \( E' \).

3.1 DEFINITION. A family \( \mathcal{F} \) of functions from \( E \) to \( E' \) is equicontinuous if for every \( \varepsilon > 0 \) and for every \( p \in E \) there exists \( \delta > 0 \) such that for all \( f \in \mathcal{F} \)

\[
d'(f(p), f(q)) < \varepsilon
\]

whenever \( d(p, q) < \delta \).

Note that the point here is that \( \delta \) depends only on \( p \) and \( \varepsilon \) but not on the function under consideration. Here is a simple result that gives you a bit of a feeling what this notion accomplishes.

3.2 THEOREM. Let \( f_n, n=1,2,3 \ldots \) be a sequence of functions from \( E \) to \( E' \) with the property that \( f_n(p) \) converges to \( f(p) \) for every \( p \in E \). Suppose further that the family \( \{f_n\}_{n=1}^\infty \) is equicontinuous. Then \( f \) is continuous, and moreover, the family \( \{f, f_1, f_2, \ldots \} \) is also equicontinuous.

Proof. Fix any \( \varepsilon \) and fix any \( p \in E \). Then there exists \( \delta > 0 \) such that whenever \( d(p, q) < \delta \), \( d'(f_n(p), f_n(q)) < \varepsilon/3 \) for all \( n = 1, 2, 3, \ldots \). Further there exists \( N \) such that both, \( d'(f(p), f_n(p)) < \varepsilon/3 \) and \( d'(f(q), f_n(q)) < \varepsilon/3 \) for all \( n > N \). Fix such a value for \( n \). Then for all \( q \) with \( d(p, q) < \delta \) we have that

\[
d'(f(p), f(q)) \leq d'(f(p), f_n(p)) + d'(f_n(p), f_n(q)) + d'(f_n(q), f(q)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .
\]
Note that since we know that whenever \( d(p, q) < \delta \), then \( d'(f_n(p), f_n(q)) < \varepsilon/3 < \varepsilon \) we know that the family \( \{f, f_1, f_2, \ldots\} \) is also an equicontinuous family.

Another simple consequence is the following

**3.3 THEOREM.** Let \( \{f_n\}_{n=1}^{\infty} \) be an equicontinuous family of functions from \( E \) to \( E' \). Assume that \( E' \) is complete and that \( f_n(p) \) converges for all \( p \in D \) where \( D \subset E \) is dense. Then \( f_n(p) \) converges for all \( p \in E \).

**Proof.** Recall that \( D \subset E \) dense means that for every \( p \in E \) and every \( \varepsilon > 0 \) there exists \( q \in D \) such that \( d(p, q) < \varepsilon \). Now pick \( p \in E \) arbitrary and pick an \( \varepsilon > 0 \). There exists \( \delta > 0 \) such that for all \( q \in E \) with \( d(p, q) < \delta \) we have for all \( n = 1, 2, 3, \ldots \) \( d'(f_n(p), f_n(q)) < \varepsilon/3 \). In particular there exists \( q \in D \) with \( d(p, q) < \delta \). Since \( f_n(q) \) converges for \( q \in D \) there exists \( N \) so that for all \( n, m > N \), \( d'(f_n(q), f_m(q)) < \varepsilon/3 \) and hence

\[
d'(f_n(p), f_m(p)) \leq d'(f_n(p), f_n(q)) + d'(f_n(q), f_m(q)) + d'(f_m(q), f_m(p)) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

Thus, \( f_n(p) \) is a Cauchy sequence in \( E' \) and since \( E' \) is complete it converges. Thus \( f_n(p) \) converges for all \( p \in E \).

If in the definition of equicontinuity, \( \delta \) does only depend on \( \varepsilon \) and not on the point \( p \in E \), then we call the family \( F \) uniformly equicontinuous. More precisely we have

**3.4 DEFINITION.** A family \( F \) of functions from \( E \) to \( E' \) is uniformly equicontinuous if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for all \( f \in F \) and all \( p, q \) with \( d(p, q) < \delta \) it follows that

\[
d'(f(p), f(q)) < \varepsilon.
\]

Here is a first interesting theorem concerning uniform equicontinuity.

**3.5 THEOREM.** Let \( \{f_n\}_{n=1}^{\infty} \) be a uniformly equicontinuous family of real valued functions on the interval \([0, 1]\). Assume further that \( f_n(x) \) converges to \( f(x) \) for all \( x \in [0, 1] \). Then the convergence is uniform.

**Proof.** Pick \( \varepsilon > 0 \). By Theorem 3.2 we know that the limiting function is continuous and that the family \( \{f, f_1, f_2, \ldots\} \) is equicontinuous. There exists \( \delta > 0 \) such that \( |f(x) - f(y)| < \varepsilon/3 \) and \( |f_n(x) - f_n(y)| < \varepsilon/3 \) for all \( n \), whenever \( |x - y| < \delta \). Now consider the points \( x_1, \ldots, x_M \) so that no point \( x \in [0, 1] \) is farther away from \( x_j \) for some \( j = 1, 2, \ldots, M \). This is a finite
set of points and hence there exists $N$, depending only on $\varepsilon$ such that for all $n > N$ and all $j = 1, \ldots, M$,

$$|f(x_j) - f_n(x_j)| < \varepsilon/3.$$ 

For any $x \in [0, 1]$ we have therefore for some $x_j$ with $|x - x_j| < \delta$ that

$$|f(x) - f_n(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)|$$

and since each term is strictly less than $\varepsilon/3$ the result follows. \hfill \Box

We are now ready to formulate and prove a central result.

**3.6 THEOREM (Arzela-Ascoli Theorem).** Let $\{f_n\}_{n=1}^\infty$ be a uniformly equicontinuous family of uniformly bounded functions on $[0, 1]$. Then there exists a subsequence $f_n(i)$ which converges uniformly on $[0, 1]$.

**Proof.** The rational numbers in $r_m \in [0, 1]$ are countable and dense. Since the functions $f_n$ are uniformly bounded we also know that $|f_n(r_m)| \leq C$ for some constant $C > 0$. From the ‘Diagonal argument’ we know that there exists a subsequence $n(i)$ such that $f_{n(i)}(r_m)$ converges for all $r_m$. By Theorem 3.3 we know that the sequence $f_{n(i)}(x), i = 1, 2, 3 \ldots$ converges for all $x \in [0, 1]$ to some function $f(x)$. By Theorem 3.2 we know that this function is continuous and by Theorem 3.5 we know that the convergence is uniform. \hfill \Box