1. Introduction

Young’s inequality in its standard form is an estimate on the $L^p$ norm of the convolution of two functions.

$$\|f * g\|_r \leq C(p, q; n) \|f\|_p \|g\|_q,$$

where $f, g$ are functions defined on $\mathbb{R}^n$ and

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)dy.$$

For dimensional reasons, the indices must be related by

$$1 + \frac{1}{r'} = 1 + \frac{1}{p} + \frac{1}{q}.$$

An equivalent version involves three functions

$$\int_{\mathbb{R}^{2n}} h(x)f(x - y)g(y)dxdy \leq C(p, q; n) \|f\|_p \|g\|_q \|h\|_r,$$

where

$$\frac{1}{r'} + \frac{1}{r} = 1.$$

Thus

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2.$$

The constant $C(p, q; n)$ denotes the sharp constant. It will depend, of course on $p, q$ and the dimension $n$.

I looked up some of the papers of W.H. Young in the Proc. London Math. Soc. 1912 and 1913, but never found the inequality quoted in that particular way, although it is proved for some particular indices.

A proof of (1) without the sharp constant is relatively easy to achieve using Hölder’s inequality. Clearly the functions $f, g, h$ can be replaced by their magnitudes. Noting that

$$\frac{p}{q'} + \frac{p}{r'} = 1 \text{ etc}$$

we

$$\int_{\mathbb{R}^{2n}} h(x)f(x - y)g(y)dxdy = \int_{\mathbb{R}^{2n}} h(x)^{q'} f(x - y)^{q'} h(x)^{p'} g(y)^{p'} f(x - y)^{p'} g(y)^{q'} dxdy$$

$$\leq \left[ \int_{\mathbb{R}^{2n}} [h(x)^{q'} f(x - y)^{q'}]^{q'} dxdy \right]^{1/q'} \left[ \int_{\mathbb{R}^{2n}} [h(x)^{p'} g(y)^{p'}]^{p'} dxdy \right]^{1/p'} \left[ \int_{\mathbb{R}^{2n}} [f(x - y)^{p'} g(y)^{p'}]^{r'} dxdy \right]^{1/r'} = \|f\|_p \|g\|_q \|h\|_r.$$
The sharp constant, including the optimizers are known for this inequality and we will concentrate on this problem for a while. The reasons for this are manifold. For once it is an interesting problem in the calculus of variations with various proofs. Immediate consequences are Nelson’s hypercontractive estimate and Gross’ log-Sobolev inequality. Further, there is an intimate connection between a generalization of Young’s inequality, the Brascamp- Lieb inequality and convex geometry. It is my aim to present these connections during the next few lectures.

The inequality can be considerably generalized. Consider $M$ functions $f_j : \mathbb{R}^{n_j} \to \mathbb{R}$, $j = 1, \ldots, M$ and linear maps $B_j : \mathbb{R}^N \to \mathbb{R}^{n_j}$. Then we may consider an inequality of the form

$$\int_{\mathbb{R}^N} \prod_{j=1}^M f_j(B_j x) d^N x \leq C \prod_{j=1}^M \|f_j\|_{L^{p_j}(\mathbb{R}^{n_j})}$$

and it is an interesting question under which circumstances the inequality holds for a finite constant $C$. This inequality was discovered by Lieb, who showed that it suffices to optimize over Gaussian functions. This non-trivial problem was then further investigated by Bennett, Carbery, Christ and Tao and also by Carlen, Lieb and Loss in the rank one case

$$\int_{\mathbb{R}^N} \prod_{j=1}^M f_j((a_j, x)) d^N x \leq C \prod_{j=1}^M \|f_j\|_{p_j}$$

where $a_j \in \mathbb{R}^n$ are given vectors. This inequality is the Brascamp-Lieb inequality. Again, one of their results is that it suffices to consider Gaussian functions when maximizing in order to compute the sharp constant. We will be more precise later.

2. Franck Barthe’s proof of Young’s inequality

Denote by $C_s$ the positive constant

$$C_s^2 = \frac{s_{1/2}}{|s'|^{1/\sigma}}$$

**Theorem 2.1.** Let $p, q, r' > 0$, $1 + \frac{1}{r'} = \frac{1}{p} + \frac{1}{q}$ and let $f, g$ be non–negative functions. If $p, q, r' \geq 1$, then

$$\|f * g\|_{r'} \leq \left( \frac{C_p C_q}{C_{r'}} \right)^n \|f\|_p \|g\|_q .$$

(Young’s inequality) If $p, q, r' \leq 1$ then

$$\|f * g\|_{r'} \geq \left( \frac{C_p C_q}{C_{r'}} \right)^n \|f\|_p \|g\|_q .$$

(Leindler-Prekopa inequality).

Moreover, there is equality in these inequalities for Gaussian functions.

**Remark 2.2.** The sharp constant in Young’s inequality has been determined independently by Beckner and Brascamp–Lieb. The sharp constant for the Leindler–Prekopa inequality has been determined by Brascamp and Lieb.
Remark 2.3. One can show that equality in the sharp Young inequality implies that the functions have to be Gaussian. This result is due to Brascamp and Lieb.

Proof of Theorem 2.1 in one dimension: We follow Barthe including the notation. According to Barthe, we rewrite the inequality slightly. Introduce

$$c = \sqrt{r/q}, \quad s = \sqrt{r/p'},$$

Note that

$$c^2 + s^2 = r\left(\frac{1}{q'} + \frac{1}{p'}\right) = 1,$$

since

$$1 + \frac{1}{r'} = \frac{1}{p} + \frac{1}{q}$$

and

$$\frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}$$

are equivalent. Moreover, $r, p', q'$ have the same sign if $p, q, r' > 1$ or $p, q, r' < 1$. Finally, neither $c$ nor $s$ vanishes if neither $r', p, q$ equals 1.

Now the two inequalities take the equivalent form

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f^{1/p}(cx - sy)g^{1/q}(sx + cy)dx\right)^{r'} dy\right)^{1/r'} \leq K(p, q) \left(\int_{\mathbb{R}} f(x)dx\right)^{1/p} \left(\int_{\mathbb{R}} g(y)dy\right)^{1/q}$$

for $p, q, r' > 1$ and the inequality is reversed if $p, q, r' < 1$. Here

$$K(p, q) = \frac{p^{1/p} q^{1/q}}{r'^{1/r'}}.$$

Remark 2.4. Note that there is equality in the above inequality if we set

$$f(x) = e^{-px^2}, \quad g(x) = e^{-qx^2}.$$

The introduction of $c, s$ comes in handy, since there is no need to work out the convolution. The Gaussians factorize.

Lemma 2.5 (Franck Barthe). Let $p, q, r' > 1$ and $1/p + 1/q = 1/r'$. Let $f, g, F, G$ be continuous positive, integrable functions such that

$$\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} F(x)dx$$

and

$$\int_{\mathbb{R}} g(x)dx = \int_{\mathbb{R}} G(x)dx.$$

Then

$$\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f^{1/p}(cx - sy)g^{1/q}(sx + cy)dx\right)^{r'} dy\right)^{1/r'} \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} F^{r'/p}(cX - sY)G^{r'/q}(sX + cY)dY\right)^{1/r'} dX.$$

(2)
Remark 2.6. Note again, that for
\[ f(x) = F(x) = e^{-px^2}, \quad g(x) = G(x) = e^{-qx^2} \]
there is equality in the above inequality.

Remark 2.7. By approximation the inequality extends also to non-negative functions that are merely integrable but not necessarily continuous or strictly positive.

Proof. Define functions \( u(t) \) and \( v(t) \) via
\[
\int_{-\infty}^{u(t)} f(x)dx = \int_{-\infty}^{t} F(x)dx \\
\int_{-\infty}^{v(t)} g(x)dx = \int_{-\infty}^{t} G(x)dx
\]
Note the relations
\[
u'(t)f(u(t)) = F(t), \quad v'(t)g(v(t)) = G(t)
\]
which show that \( u', v' \) are positive.

Now there exists a function \( h(y) \) with \( \int_{\mathbb{R}} h(y)\gamma dy = 1 \) so that
\[
\left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f^{1/p}(cx - sy)g^{1/q}(sx + cy)dx \right)^{r'} dy \right)^{1/r'} = \int_{\mathbb{R}^2} f^{1/p}(cx - sy)g^{1/q}(sx + cy)h(y)dxdy.
\]
Next, we change variables
\[
x = cu(cX - sY) + sv(sX + cY), \quad y = -su(cX - sY) + cv(sX + cY)
\]
which has the Jacobian
\[
dx\,dy = u'(cX - sY)v'(sX + cY)dX\,dY.
\]
Therefore
\[
\int_{\mathbb{R}^2} f^{1/p}(cx - sy)g^{1/q}(sx + cy)h(y)dxdy
\]
\[
= \int_{\mathbb{R}^2} f^{1/p}(u(cX - sY))g^{1/q}(v(sX + cY))h(-su(cX - sY) + cv(sX + cY))u'(cX - sY)v'(sX + cY)dX\,dY
\]
\[
= \int_{\mathbb{R}^2} F^{1/p}(cX - sY)G^{1/q}(sX + cY)h(-su(cX - sY) + cv(sX + cY))u'(cX - sY)^{1/r'}v'(sX + cY)^{1/q'}dX\,dY.
\]
Hölder’s inequality yields the upper bound
\[
\left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} F^{r'/p}(cX - sY)G^{r'/q}(sX + cY)dY \right)^{1/r'} \times \\
\left( \int_{\mathbb{R}} h^{r}(-su(cX - sY) + cv(sX + cY))u'(cX - sY)^{r'/p}v'(sX + cY)^{r'/q}dY \right)^{1/r} \right)^{1/r} \,dX.
\]
Note that
\[
u'(cX - sY)^{r'/p}v'(sX + cY)^{r'/q} = u'(cX - sY)^{s^2}v'(sX + cY)^{c^2} \leq s^2u'(cX - sY) + c^2v'(sX + cY)
\]
by the arithmetic-geometric mean inequality. Note: For \( A, B > 0 \), the concavity of the log function implies
\[
\log(\alpha A + (1 - \alpha)B) \geq \alpha \log(A) + (1 - \alpha) \log B = \log(A^\alpha B^{1-\alpha}).
\]
If we set
\[ a(X, Y) = -su(cX - sY) + cv(sX + cY) \]
we find that
\[ \frac{\partial a}{\partial Y}(X, Y) = s^2u'(cX - sY) + c^2v'(sX + cY) . \]
Thus,
\[
\int_{\mathbb{R}} h'(-su(cX - sY) + cv(sX + cY))u'(cX - sY) v'(sX + cY)\,dY \\
= \int_{\mathbb{R}} h'(a(X, Y))u'(cX - sY) v'(sX + cY)\,dY \\
\leq \int_{\mathbb{R}} h'(a(X, Y)) \frac{\partial a}{\partial Y}(X, Y)\,dY = \int_{\mathbb{R}} h'(x)\,dx = 1 .
\]

**Remark 2.8.** Young’s inequality in one dimension is now an immediate consequence of Barthe’s lemma. The reverse Young is also not difficult but we refrain from spelling out the argument, since it will not be needed in the following.

### 3. Moving from One to Higher Dimensions: The Tensorial Property

Recall Minkowski’s inequality. We follow Lieb and Loss.

**Lemma 3.1.** Let \( \Omega \) and \( \Gamma \) be two sigma finite measure spaces with measures \( \mu \) and \( \nu \) respectively. Let \( f \) be a non-negative function on \( \Omega \times \Gamma \) which is measurable (with respect to the product sigma algebra) and let \( 1 \leq p < \infty \). Then
\[
\left( \int_{\Omega} \left( \int_{\Gamma} f(x, y)^p \mu(dx) \right)^{1/p} \nu(dy) \right) \geq \left( \int_{\Omega} \left( \int_{\Gamma} f(x, y)^p \nu(dy) \right)^{1/p} \mu(dx) \right)^{1/p} .
\]
Equality and finiteness in (3) for \( 1 < p < \infty \) imply the existence of a \( \mu \) measurable function \( \alpha : \Omega \to \mathbb{R}^+ \) and a \( \nu \) measurable function \( \beta : \Gamma \to \mathbb{R}^+ \) such that for \( \mu \times \nu \) almost every \( (x, y) \)
\[ f(x, y) = \alpha(x)\beta(y) . \]

**Proof.** Write
\[ H(x) := \int_{\Gamma} f(x, y)\mu(dy) \]
so that the \( p \)-th power of the right side of (3), using Fubini’s theorem, can be written as
\[ \int_{\Omega} H(x)^p \mu(dx) = \int_{\Omega} \int_{\Gamma} H(x)^{p-1} f(x, y)\nu(dy)\mu(dx) \]
which, by Hölder’s inequality leads to
\[ \int_{\Omega} H(x)^p \mu(dx) \leq \int_{\Gamma} \left( f(x, y)^p \mu(dx) \right)^{1/p} \left( \int_{\Omega} H(x)^p \mu(dx) \right)^{p-1} \nu(dy) \]
which, upon dividing by the last fact yields (3). Now we chase the meaning of the equality sign in the use of Hölder’s inequality. For \( \nu \) a.e. every \( y \) there exists \( \lambda(y) \) independent of \( x \) so that
\[ f(x, y) = \lambda(y)H(x) . \]
The fact that $\lambda(y)$ is measurable follows from
\[
\lambda(y) \int_{\Omega} H(x)^p \mu(dx) = \int_{\Omega} f(x,y)^p \mu(dx).
\]

**Lemma 3.2** (Segal’s Lemma). We have that
\[
C(p, q; n + m)) = C(p, q; n)C(p, q; m).
\]

**Proof.** By using approximate trial functions it is easy to see that
\[
C(p, q; n + m)) \geq C(p, q; n)C(p, q; m).
\]
Now for the converse, let $f(x_1, x_2), g(x_1, x_2)$ be two functions in $L^p(\mathbb{R}^{n+m}), L^q(\mathbb{R}^{n+m})$ respectively. Here $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^m$. Now
\[
\|f \ast g\|_{L^r(\mathbb{R}^{n+m})} = \left( \int dx_1 \int dx_2 \int dy_1 \int dy_2 f(x_1 - y_1, x_2 - y_2)g(y_1, y_2)^r \right)^{1/r'}
\]
\[
= \left( \int dx_1 \left[ \left( \int dx_2 \int dy_1 \int dy_2 f(x_1 - y_1, x_2 - y_2)g(y_1, y_2)^r \right)^{1/r'} \right]^{r'} \right)^{1/r'}
\]
which, by Minkowski’s inequality,
\[
\leq \left( \int dx_1 \left[ \int dy_1 \left( \int dx_2 |f(x_1 - y_1, \cdot)g(y_1, \cdot)|^{r'} \right)^{1/r} \right]^{r'} \right)^{1/r'}
\]
Using Young’s inequality, this is bounded above by
\[
C(p, q; m) \left( \int dx_1 \left[ \int dy_1 \|f(x_1 - y_1, \cdot)\|_{L^p(\mathbb{R}^m)} \|g(y_1, \cdot)\|_{L^q(\mathbb{R}^m)} \right]^{r'} \right)^{1/r'}
\]
which, once more by Young’s inequality, leads to the upper bound
\[
C(p, q; m)C(p, q; n)\|f\|_{L^p(\mathbb{R}^{n+m})}\|g\|_{L^p(\mathbb{R}^{n+m})}.
\]
This proves the Lemma. □