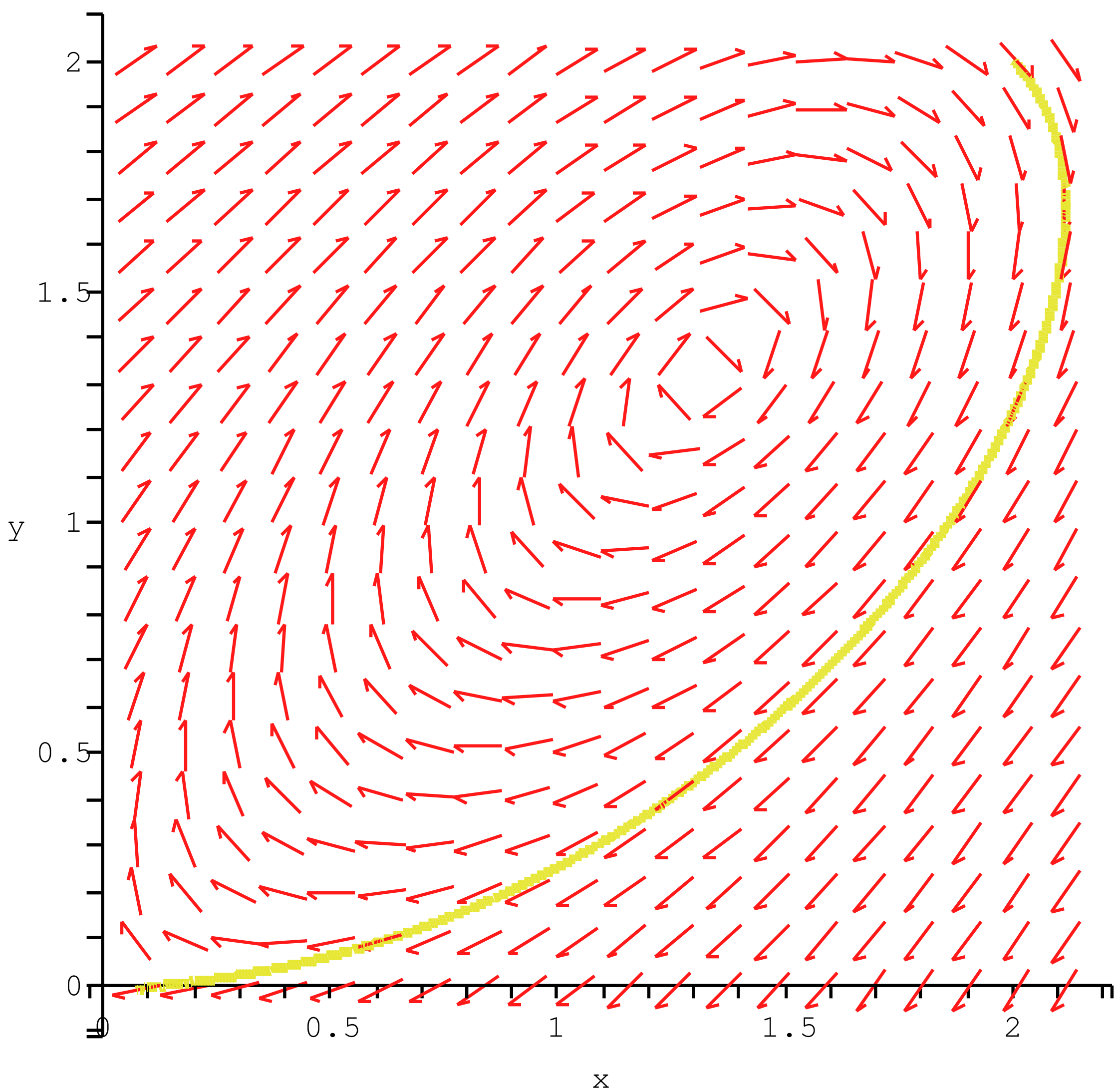
Solutions of selected problems of chapter 5

Section 4:

Problem 1: a) Critical points: $(0,0)$, $(4/3, 4/3)$. The Hessian at $(0,0)$ has the eigenvalues 4 and -4 and hence $(0,0)$ is unstable. At $(4/3, 4/3)$ the Hessian has the eigenvalues 12 and 4. Since the gradient flow is given by

$$\mathbf{x}'(t) = -\nabla f(\mathbf{x}(t)) \ .$$

Hence the critical point $(4/3, 4/3)$ is stable.
b) Forward Euler, $0 \leq t \leq 1$, stepsize 0.0001. Maple command: phaseportrait($[D(x)(t) = 3*(y(t))^2 - 4*x(t), D(y)(t) = -3*(x(t))^2 + 4*y(t)], [x(t), y(t)], t = 0..1, [[x(0) = 2, y(0) = 2]]$, stepsize = 0.0001, method = classical[foreuler]);



c) Calculating

$$|\nabla \cdot \frac{\nabla f}{|\nabla f|}| = |\partial_x \left(\frac{f_x}{\sqrt{f_x^2 + f_y^2}} \right) + \partial_y \left(\frac{f_y}{\sqrt{f_x^2 + f_y^2}} \right) |$$

at the point $(2, 2)$ yields $4/\sqrt{2}$ for the curvature.

Problem 4: The stability of the perpendicular gradient flow depends on the eigenvalues of the matrix

$$\begin{bmatrix} -f_{xy} & f_{yy} \\ f_{xx} & f_{xy} \end{bmatrix} .$$

Since the trace vanishes, the eigenvalues must be of opposite sign and hence the equilibrium points of the flow are not stable.

Section 5:

Problem 1: Using

$$\cos(\theta) = \frac{\mathrm{Tr}Q - 1}{2}$$

one finds $\cos(\theta_1) = -8/9$ and hence $\theta_1 \approx 2.6657$. Rotation axis is along the vector $(3, 2, 2)$ and hence the plane of rotation is $3x + 2y + 2z = 0$.

Problem 3: $\cos(\theta_2) = 22/45$. $\theta_2 \approx 1.05998$, i.e., close to $\pi/3$. Axis of rotation: $(-2, 2, -1)$. Plane of rotation: $-2x + 2y - z = 0$.

Problem 5: Denote the rows of $Q - I$ by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. If $\mathbf{e}$ denotes the axis of rotation we have that

$$\mathbf{a}_1 \cdot \mathbf{e} = 0 \ , \ \mathbf{a}_2 \cdot \mathbf{e} = 0 \ , \ \mathbf{a}_3 \cdot \mathbf{e} = 0 \ .$$

Hence the three vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are in the plane of rotation and the cross product of any two non proportional of these vectors yields a multiple of the axis of rotation.

Problem 7: $Q(-\mathbf{u}, \theta) = Q(\mathbf{u}, 2\pi - \theta)$. We know that

$$Q(\mathbf{u}, \theta) = U \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} U^t \ ,$$

where U is a rotation independent of θ. From this it follows that $Q(\mathbf{u}, \theta)Q(\mathbf{u}, 2\pi - \theta) = Q(\mathbf{u}, 2\pi) = I$.

Problem 9: Again, we know that

$$Q(\mathbf{u}, \theta) = U \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} U^t \ ,$$

and hence

$$Q(\mathbf{u}, \theta/2) = U \begin{bmatrix} \cos(\theta/2) & -\sin(\theta/2) & 0 \\ \sin(\theta/2) & \cos(\theta/2) & 0 \\ 0 & 0 & 1 \end{bmatrix} U^t \ ,$$

has the property that $Q(\mathbf{u}, \theta/2)^2 = Q(\mathbf{u}, \theta)$.

Problem 11: By a reflection we mean an orthogonal matrix P that has the property that $P^2 = I$. If the determinant of a 2×2 orthogonal matricx is -1, it is a reflection. This is not the case for 3×3 orthogonal matrices. Here is a counterexample. If Q is an orthogonal matrix with determinant $+1$, it can be written in the form

$$Q = U \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} U^t \ ,$$

for some angle θ. Now, $(-I)Q$ is an orthogonal matrix with determinant -1 and its square is

$$Q^2 = U \begin{bmatrix} \cos(2\theta) & -\sin(2\theta) & 0 \\ \sin(2\theta) & \cos(2\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} U^t \ .$$

which in general is not the identity matrix.

Section 6: COMING SOON!