Chapter 5, Section 6:

Problem 1: $Q_2 = UQ_1$ and hence

$$U = Q_2 Q_1^t = \begin{bmatrix} -71/75 & -22/75 & 2/15 \\ 22/75 & -46/75 & 11/15 \\ -2/15 & 11/15 & 2/3 \end{bmatrix}$$

Further

$$\cos(\theta) = -71/75 \ ,$$

or

$$\theta \approx 2.813524772 \ .$$

The axis of rotation is given by

$$2\sin(\theta)\mathbf{u} = \begin{bmatrix} 0 \\ 4/15 \\ 44/75 \end{bmatrix} \ ,$$

and hence

$$\mathbf{u} = \frac{1}{\sqrt{146}} \begin{bmatrix} 0 \\ 5 \\ 11 \end{bmatrix} \ .$$

The time dependent mrotation is then given by

$$U(t) = \cos(t\theta)I + (1 - \cos(t\theta))\frac{1}{146} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 25 & 55 \\ 0 & 55 & 121 \end{bmatrix} + \sin(t\theta)\frac{1}{\sqrt{146}} \begin{bmatrix} 0 & -11 & 5 \\ 11 & 0 & 0 \\ -5 & 0 & 0 \end{bmatrix} \ .$$

Problem 3:
$Q_1^2 = Q_1 U$ i.e., $U = Q_1$.

$$\cos(\theta) = -8/9 \ ,$$

$$2\sin(\theta)\mathbf{u} = \begin{bmatrix} 2/3 \\ 4/9 \\ 4/9 \end{bmatrix} \ ,$$

and hence

$$\mathbf{u} = \frac{1}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \ .$$

Thus the time dependent rotation is given by

$$U(t) = \cos(t\theta)I + (1 - \cos(t\theta))\frac{1}{17} \begin{bmatrix} 9 & 6 & 6 \\ 6 & 4 & 4 \\ 6 & 4 & 4 \end{bmatrix} + \sin(t\theta)\frac{1}{\sqrt{17}} \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} \ .$$

Problem 5: Find all B antisymmetric so that

$$e^B = Q_1 \ .$$

From problem 3 we know that

$$B_{\mathbf{u}} = \frac{1}{\sqrt{17}} \begin{bmatrix} 0 & -2 & 2 \\ 2 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

works in the sense that

$$e^{\theta B_{\mathbf{u}}} = Q_1$$

where $\cos(\theta) = -8/9$. Thus the matrices

$$B_n = (\theta + 2\pi n) B_{\mathbf{u}} \ ,$$

for any integer n do the job.

Problem 7: Find all B antisymmetric so that

$$e^B = Q_3 \ .$$

The angle of rotation for Q_3 is given by $\cos(\theta) = 3/5$. Moreover

$$2\sin(\theta) B_{\mathbf{u}} = \begin{bmatrix} 0 & 8/15 & 16/15 \\ -8/15 & 0 & 16/15 \\ -16/15 & -16/15 & 0 \end{bmatrix} \ .$$

Since $\sin(\theta) = 4/5$ we get that

$$B_{\mathbf{u}} = \frac{1}{3} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$$

Again

$$(\theta + 2\pi n) B_{\mathbf{u}}$$

does the job.

Problem 9: Note that

$$B_{\mathbf{u}} \mathbf{x} = \mathbf{u} \times \mathbf{x} \ .$$

Now

$$Q B_{\mathbf{u}} Q^t \mathbf{x} = Q(\mathbf{u} \times Q^t \mathbf{x}) = Q\mathbf{u} \times Q Q^t \mathbf{x} = Q\mathbf{u} \times \mathbf{x} = B_{Q\mathbf{u}} \mathbf{x} \ .$$

Hence

$$Q B_{\mathbf{u}} Q^t = B_{Q\mathbf{u}} \ .$$