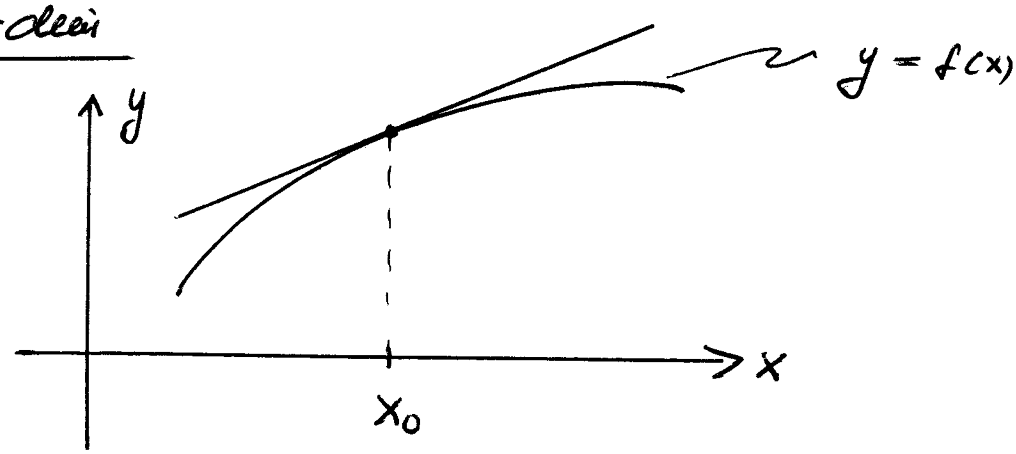
Tangent planes and gradients

1-dim



Line tangent to the graph of f at x_0.

Slope: $f'(x_0)$ the derivative of f at x_0

Equation for tangent line

$$y = a + f'(x_0)(x - x_0)$$

at $x = x_0$ $\quad y = f(x_0) \Rightarrow a = f(x_0)$

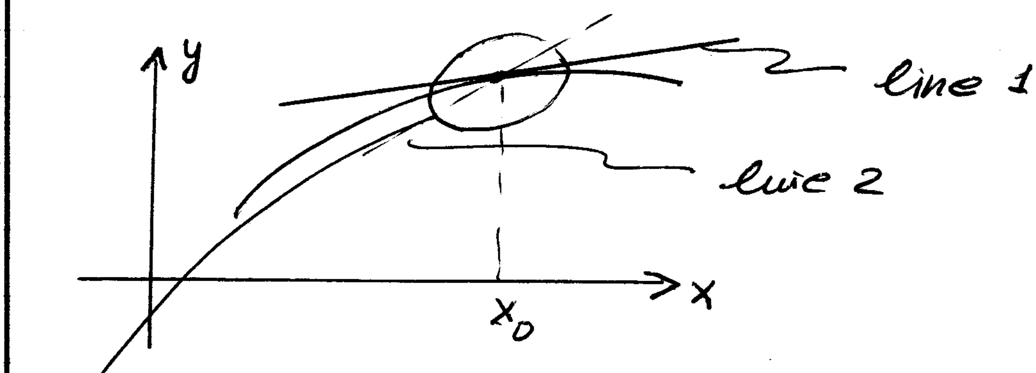
$$\underline{y = f(x_0) + f'(x_0)(x - x_0)}$$

We have tacitly assumed that $f(x)$ is a differentiable function at x_0, i.e.,

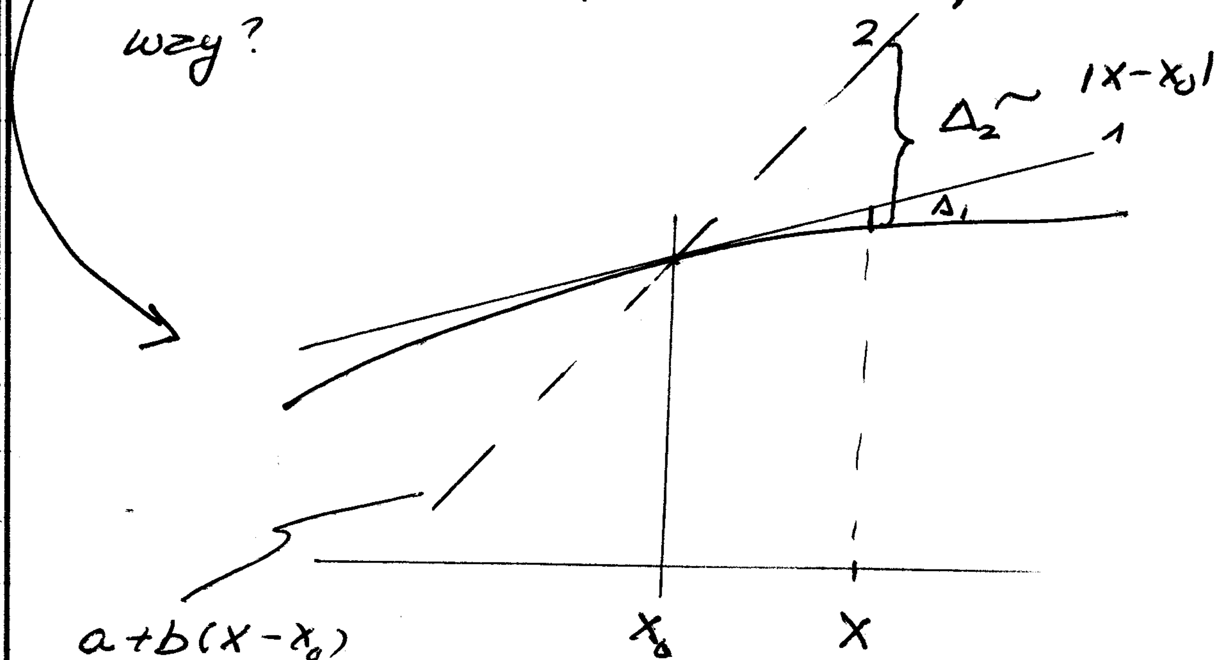
$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{exists}.$$

Slightly diff. point of view

Best linear approximation



What is the difference between line 1 and line 2 expressed in a quantitative way?



$\Delta_1 \to 0$ faster than Δ_2

as $x \to x_0$

and hence faster than $|x-x_0|$.

Definition

$f(x)$ is differentiable at x_0
if there exists a linear function
$a + b(x - x_0)$
such that

$$\frac{|f(x) - a - b(x - x_0)|}{|x - x_0|} \to 0$$

as $x \to x_0$

Clearly $|f(x) - a - b(x - x_0)| \to 0$ as $x \to x_0$

$\Rightarrow a = f(x_0)$

$$\left| \frac{f(x) - f(x_0) - b(x - x_0)}{x - x_0} \right| \to 0 \quad \text{as } x \to x_0$$

$\Downarrow$

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - b \right| \to 0 \quad \text{as } x \to x_0$$

$\Rightarrow b = f'(x_0)$.

Hence the linear function is given by
$f(x_0) + f'(x_0)(x - x_0)$ ∇.

$f(x,y)$, function of two variables.

Definition

f is diff. at $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \vec{x}_0$ if there ex

$\exists$ linear function

$$a + b(x - x_0) + c(y - y_0)$$

such that

$$\frac{|f(x,y) - a - b(x-x_0) - c(y-y_0)|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} \longrightarrow 0$$

as $\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ i.e.

as $|\vec{x} - \vec{x}_0| \longrightarrow 0$!

Clearly

$$|f(x,y) - a - b(x-x_0) - c(y-y_0)| \longrightarrow 0$$

as $\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

$\Longrightarrow \quad a = f(x_0, y_0)$.

<u>Special case</u> Fix $y = y_0$, $x \neq x_0$ variable

$$\frac{|f(x,y_0) - f(x_0,y_0) - b(x-x_0)|}{|x-x_0|} \to 0 \quad \text{as } x \to x_0$$

$$\left| \frac{f(x,y_0) - f(x_0,y_0)}{x-x_0} - b \right| \to 0 \quad \text{as } x \to x_0$$

$$\Rightarrow \quad b = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x-x_0} = \frac{\partial f}{\partial x}(x_0,y_0)$$

similary $\displaystyle c = \lim_{y \to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{y-y_0} = \frac{\partial f}{\partial y}(x_0,y_0).$

Hence the best linear approximation at $\overline{x}_0$ is given by the linear function

$$f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ are called <u>partial</u> derivatives

Examples

1) $f(x,y) = \cos(xy)$ $\qquad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \pi \end{bmatrix}.$

Find the best linear approximation.

$$\cos(x_0 y_0) = \cos \pi = -1$$

$$\cancel{\frac{\partial}{\partial x} \cos(x_0, y} \quad \frac{\partial f}{\partial x}(x_0, y_0) = -y_0 \sin(x_0 y_0)$$

$$= -\pi \sin(\pi) = 0.$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = -x_0 \sin(x_0 y_0) = -\pi \sin \pi = 0$$

The best linear approximation of $f(x,y)$ at ~~$x_0, y_0 =$~~ $\begin{bmatrix} 1 \\ \pi \end{bmatrix}$ is the constant function

$$-1 \quad //.$$

2) $f(x,y) = e^{-x^2+xy}$ $\quad$ a $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$$f(1,2) = e^{-1+2} = e$$

$$\cancel{\frac{\partial f}{\partial x} = (e^{-x^2+xy})} \quad \frac{\partial f}{\partial x}(1,2) = \frac{d}{dx} e^{-x^2+2x} \Big|_{x=1}$$

$$= e^{-x^2+2x}(-2x+2)\Big|_{x=1} = 0$$

$$\frac{\partial f}{\partial y}(1,2) = \frac{d}{dy}\, e^{-1+y}\Big|_{y=2}$$

$$= e^{-1+y}\Big|_{y=2}\, \frac{d}{dy}(-1+y) = e\,.$$

Best linear approximation:

$$e + e(y-2)$$

Funktion of three variables

$f(x,y,z)$

Best linear approximation at

$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$.

$$f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(x_0,y_0,z_0)\,(x-x_0)$$
$$+ \frac{\partial f}{\partial y}(x_0,y_0,z_0)\,(y-y_0)$$
$$+ \frac{\partial f}{\partial z}(x_0,y_0,z_0)\,(z-z_0)$$

We denote

$$\nabla f(\overline{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}_0) \\ \frac{\partial f}{\partial y}(\overline{x}_0) \end{bmatrix}$$

or in three varible

$$\nabla f(\overline{x}_0) = \begin{bmatrix} \frac{\partial f}{\partial x}(\overline{x}_0) \\ \frac{\partial f}{\partial y}(\overline{x}_0) \\ \frac{\partial f}{\partial z}(\overline{x}_0) \end{bmatrix}$$

and call it the gradient.

Best linear fit: $f(\overline{x}_0) + \nabla f(\overline{x}_0)\cdot(\overrightarrow{x} - \overline{x}_0)$

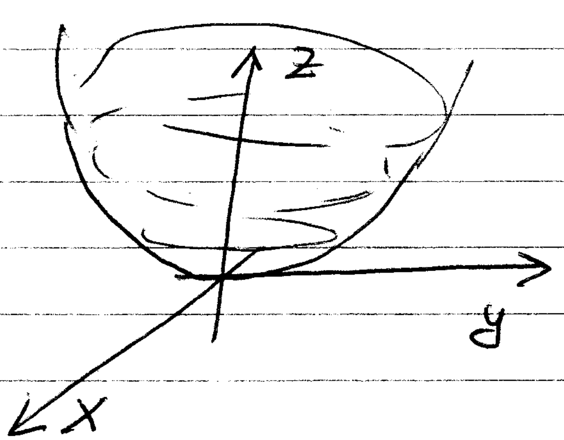
Problem:

Find the plane, tangent to the surface

$$z = x^2+y^2$$

at the point $\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$.

Note $5 = 4+1 = 2^2+1^2$



Think of

$$f(x,y) = x^2+y^2$$

~~Find the best line~~

$z = x^2+y^2$ is the graph of x^2+y^2.

The best linear fit at $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$f(2,1) + \nabla f(2,1) \cdot \begin{bmatrix} (x-2) \\ (y-1) \end{bmatrix}.$$

$$f(2,1) = 2^2+1 = 5$$

$$\nabla f(x,y)\Big|_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}\Big|_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$5 + 4(x-2) + 2(y-1)$$

$$z = 5 + 4(x-2) + 2(y-1)$$

is the graph of the linear function

$$5 + 4(x-2) + 2(y-1).$$

The graph is a plane which is tangent to $z = x^2+y^2$ at $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Problem

Find ~~the~~ vector $\neq 0$ that is normal to the surface

$$z = f(x, y)$$

at the ~~point~~ $\bar{x}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.

$$z = f(\bar{x}_0) + \nabla f(\bar{x}_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Equation for the tangent plane

$$z - \frac{\partial f}{\partial x}(\bar{x}_0)x - \frac{\partial f}{\partial y}(\bar{x}_0)y = \underbrace{f(\bar{x}_0) - \nabla f(\bar{x}_0) \cdot \bar{x}_0}$$

$$\begin{bmatrix} -\frac{\partial f}{\partial x}(\bar{x}_0) \\ -\frac{\partial f}{\partial y}(\bar{x}_0) \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \text{const}$$

Normal vector: $\begin{bmatrix} \nabla f(\bar{x}_0) \\ -1 \end{bmatrix}$

or $\begin{bmatrix} -\nabla f(\bar{x}_0) \\ 1 \end{bmatrix}$.

Example

$$f(x,y) = \sin(x^2+y)$$

$$\overline{x}_0 = \begin{bmatrix} \sqrt{\pi} \\ \pi \end{bmatrix}.$$

Normal vector to the surface

$$z = \sin(x^2+y) \quad \text{at} \quad \overline{x}_0 = \begin{bmatrix} \sqrt{\pi} \\ \pi \end{bmatrix}.$$

$$\nabla f(\overline{x}_0) = \cos(x^2+y) \begin{bmatrix} 2x \\ 1 \end{bmatrix}$$

$$\nabla f(\overline{x}_0) = \cos(\pi+\pi) \begin{bmatrix} 2\sqrt{\pi} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{\pi} \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 2\sqrt{\pi} \\ 1 \\ -1 \end{bmatrix}$$ Normal vector.