

Chapter 0 of Calculus⁺⁺, Differential calculus with several variables

Background material

by

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Table of Contents

Section 1: Lines and planes in \mathbb{R}^3

1.1 The description of lines and planes in \mathbb{R}^3 by equations	0-3
1.2 The parametric description of lines and planes in \mathbb{R}^3	0-5
1.3 Recovering equations from a parameterization	0-7
1.4 Lines intersecting planes	0-11
1.5 Lines reflecting off planes	0-15
1.6 Distance problems	0-17

Section 2: The distance between two lines in \mathbb{R}^3

2.1 An algebraic point of view	0-22
2.2 A differential geometric point of view	0-23
2.3 Comparing the two points of view.....	0-24

Section 3: Functions from \mathbb{R} to \mathbb{R}^n

3.1 Curves in \mathbb{R}^n	0-26
3.2 Velocity and acceleration	0-28

Section 1: Lines and planes in \mathbb{R}^3

We have two ways of describing lines and planes in \mathbb{R}^3 : We can specify them in terms of *equations*, or in *parametric form*. Both are useful for answering various geometric questions about lines and planes, and it is important to be able to pass back and forth between them.

1.1: The description of lines and planes in \mathbb{R}^3 by equations

A plane in \mathbb{R}^3 is the solution of a single linear equation of the form

$$\mathbf{a} \cdot \mathbf{x} = d \quad (1.1)$$

where $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. If for example, $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and $d = 4$, this becomes

$$x + 2y + 3z = 4. \quad (1.2)$$

There is an even more geometric way to write (1.1). Let \mathbf{x}_0 be any particular point on the plane; i.e., any particular solution of (1.2). Finding particular solutions is easy: Just set all but one variable equal to zero. With $y = z = 0$, (1.2) becomes $x = 4$, so one

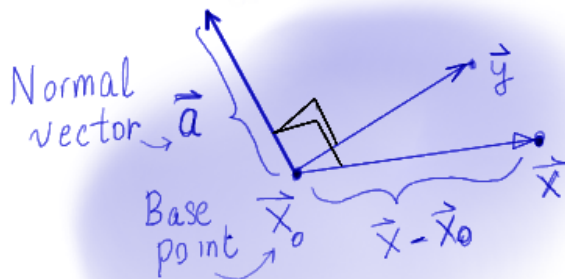
particular solution is $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$. You can check that indeed, $\mathbf{a} \cdot \mathbf{x}_0 = d = 4$.

In general, if \mathbf{x}_0 is any particular solution of (1.1), then (1.1) is equivalent to

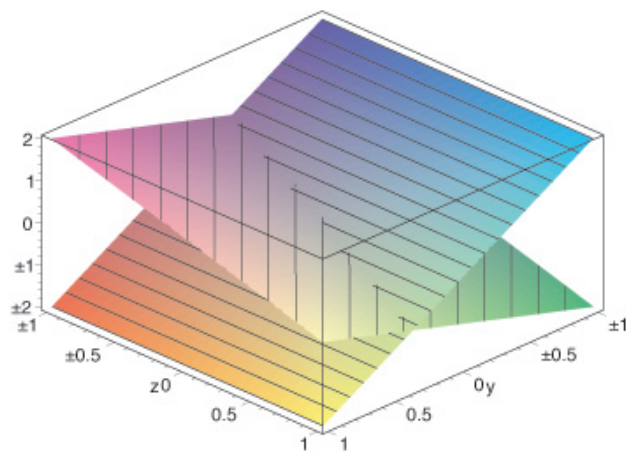
$$\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0. \quad (1.3)$$

This is the *normal vector and base point* for of the equation for a plane.

Here is an illustration showing a patch of a plane. You see the base point \mathbf{x}_0 and the normal vector \mathbf{a} attached to \mathbf{x}_0 . Also indicated are two points \mathbf{x} and \mathbf{y} lying in the plane. You see that \mathbf{a} and $\mathbf{x} - \mathbf{x}_0$, the vector running from \mathbf{x}_0 to \mathbf{x} , are orthogonal, and that this characterizes membership in the plane.



A line in R^3 is the intersection of two planes, as illustrated below:



Therefore, we can specify a line as the solution set of a system of two equations specifying planes; i.e., two equations of the form (1.1):

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= d_1 \\ \mathbf{a}_2 \cdot \mathbf{x} &= d_2 \end{aligned} \tag{1.4}$$

Introduce the 2×3 matrix A whose first row is \mathbf{a}_1 , and whose second row is \mathbf{a}_2 ; i.e., $A = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$. Then by a fundamental formula for vector–matrix multiplication,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{x} \\ \mathbf{a}_2 \cdot \mathbf{x} \end{bmatrix},$$

and so if we introduce the vector $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$, we have that (1.4) is equivalent to the matrix equation

$$A\mathbf{x} = \mathbf{d}. \tag{1.5}$$

For example, consider the case

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{1.6}$$

Then the matrix equation $A\mathbf{x} = \mathbf{d}$ is equivalent to the system of equations

$$\begin{aligned} x + 2y + 3z &= 1 \\ 3x + 2y + z &= -1 \end{aligned} \tag{1.7}$$

Its solution set is clearly the line formed by the intersection of the two planes

$$x + 2y + 3z = 1 \quad \text{and} \quad 3x + 2y + z = -1. \quad (1.8)$$

That summarizes the description in terms of equations. What is the parametric form of description?

1.2: The parametric description of lines and planes in R^3

• *The parametric description is what we get when we solve the equations.*

Let's take the equations for a line using A and \mathbf{d} from (1.6). To solve this equation, you row reduce $[A|\mathbf{d}] = \begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 3 & 2 & 1 & | & -1 \end{bmatrix}$ which leads to $\begin{bmatrix} 1 & 2 & 3 & | & 1 \\ 0 & 1 & 2 & | & 1 \end{bmatrix}$.

The variable z is non-pivotal, and from the second equation we see $y + 2z = 1$, or $y = 1 - 2z$. The first equation then becomes $x + 2(1 - 2z) + 3z = 1$, or $x = -1 + z$.

Plugging these results into $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get $\begin{bmatrix} -1 + z \\ 1 - 2z \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

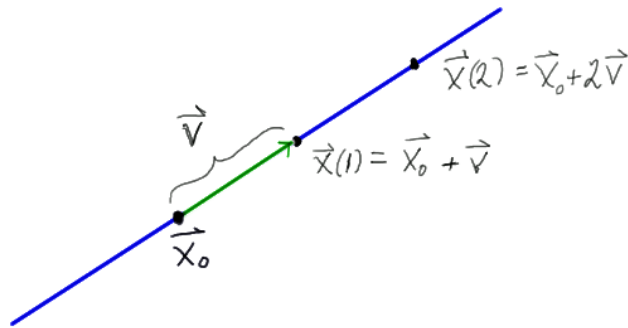
Defining $\mathbf{x}_0 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, we have that the solution set is exactly the set

of points $\mathbf{x}(t)$ where t ranges over the real numbers, and

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}. \quad (1.9)$$

Notice that we have changed the name of the parameter from z to t . This is not necessary, but it signals a shift in the way we think about z – as a parameter now instead of a variable.

Every line may be parameterized as in (1.9) for some vector \mathbf{x}_0 , called a *base point* and some non-zero vector \mathbf{v} called a *direction vector*, and this is a parametric description of the line. Here is an illustration of the scheme:



As we have seen, one way to find the direction vector \mathbf{v} given the system of equations for the line is to solve the system of equations. There is an alternative involving the cross product. Here is how it works for the line given by (1.7):

A direction vector of this line is some non-zero vector that is orthogonal to the normal vectors to both of the planes specified in (1.7). The normal vectors to these two planes are

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} .$$

One way to get a vector orthogonal to both of these vectors is to take their cross product; that is we can use

$$\mathbf{v} = \mathbf{a}_1 \times \mathbf{a}_2$$

as the direction vector. Computing this we find $\mathbf{v} = \mathbf{a}_1 \times \mathbf{a}_2 = \begin{bmatrix} -4 \\ 8 \\ -4 \end{bmatrix}$.

To find a base point, set $z = 0$ in (1.8), which then become the simple system

$$x + 2y = 1$$

$$3x + 2y = -1 .$$

This is easily solved, with the result that $x = -1$, and $y = 1$. Hence we have the base point

$$\mathbf{x}_0 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} .$$

This gives us the parameterization $\mathbf{x}(t) = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 8 \\ -4 \end{bmatrix} = \begin{bmatrix} -1 - 4t \\ 1 + 8t \\ -4t \end{bmatrix}$. This is not

the same parameterization we found before, but it does parameterize the same line.

The cross product approach is favored by many textbooks, although you still have to solve a system of equations to find the base point. The linear algebra approach is probably simpler to apply. *In any case, keep in mind that finding a parameterization starting from the equations just means solving the equations.*

To find the parametric description of a plane, we again just solve the equation. For example, consider the plane given by

$$x + 2y + z = 4$$

as above. We do not need to bring in any matrices and row reduction here – there would be just one row since there is just one equation. Instead, we just solve for x in terms of y

and z , and find $x = 4 - 2y - z$. Plugging this into $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get

$$\mathbf{x} = \begin{bmatrix} 4 - 2y - z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

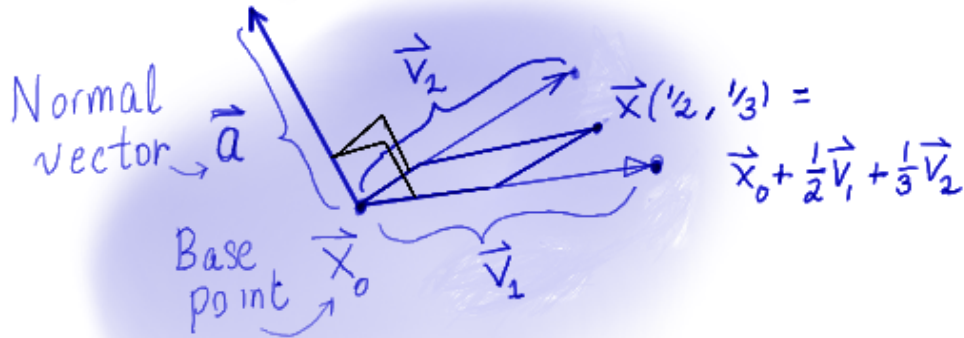
We could use y and z as the parameters, but let's change their names to s and t , so we have that the plane is parameterized by

$$\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$$

where

$$\mathbf{x}_0 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} .$$

Here is an illustration of the scheme:



You see the vector $\mathbf{x}(1/2, 1/3) = \mathbf{x}_0 + (1/2)\mathbf{v}_1 + (1/3)\mathbf{v}_2$, and you can see how any point in the plane can be reached by starting from \mathbf{x}_0 , moving in the direction of \mathbf{v}_1 to $\mathbf{x}_0 + s\mathbf{v}_1$, for some s , and from there moving in the direction \mathbf{v}_2 to $\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for some t . Two linearly independent direction vectors suffice to reach every point in the plane, and the values of s and t are uniquely determined.

1.3 Recovering equations from a parameterization

Sometimes a parameterization is easy to find, and what would be most useful to us would be an equation.

For example, suppose we are given 4 points

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{p}_3 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_4 = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} , \quad (1.10)$$

and we are asked:

Do all of these points lie in the same plane?

To answer this question, consider the plane determined by the first three points. It is easy to parameterize this: Use \mathbf{p}_1 as the base point, and get the direction vectors from the

$\mathbf{p}_2 - \mathbf{p}_1$ and $\mathbf{p}_3 - \mathbf{p}_1$. Hence we define

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} . \quad (1.11)$$

Then the plane is parameterized by

$$\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2 = \begin{bmatrix} 1 + 2s \\ 2 + t \\ 3 - 2s - t \end{bmatrix} . \quad (1.12)$$

To find the equation of the plane in the form $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, all we need to do is to find the normal vector \mathbf{a} , since we already have the base point \mathbf{x}_0 . A valid choice for this is

$$\mathbf{a} = \mathbf{v}_1 \times \mathbf{v}_2$$

Indeed, since this vector is orthogonal to both \mathbf{v}_1 and \mathbf{v}_2 , we have

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2) &= \mathbf{a} \cdot \mathbf{x}_0 + s\mathbf{a} \cdot \mathbf{v}_1 + t\mathbf{a} \cdot \mathbf{v}_2 \\ &= \mathbf{a} \cdot \mathbf{x}_0 \end{aligned}$$

for all s and t . So when $\mathbf{x} = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for some s and t , $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. We summarize:

• *Given a plane in parametric form $\mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$, one can get an equation for this plane in the form $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ by using the same base point \mathbf{x}_0 and taking*

$$\mathbf{a} = \mathbf{v}_1 \times \mathbf{v}_2 .$$

For example, consider the case in which \mathbf{x}_0 , \mathbf{v}_1 and \mathbf{v}_2 are given by (1.11). Then

$$\mathbf{a} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} .$$

We also compute

$$\mathbf{a} \cdot \mathbf{x} = 2x + 2y + 2z \quad \text{and} \quad \mathbf{a} \cdot \mathbf{x}_0 = 12$$

so the equation $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$ written out in terms of x , y and z is

$$2x + 2y + 2z = 12 ,$$

or, what is the same thing,

$$x + y + z = 6 . \quad (1.13)$$

This is the equation for the plane passing through the first three points in the list (1.10). You can easily check that these points do satisfy the equation. For example, with \mathbf{p}_1 , $x = 1$, $y = 2$ and $z = 3$. With these values of x , y and z , (1.13) is clearly satisfied.

We can now easily decide whether \mathbf{p}_4 lies in the same plane as the first three points. With $x = 4$, $y = -1$ and $z = 3$, the equation (1.13) is satisfied, so it is in the plane.

That takes care of going from a parameterization to an equation for planes in R^3 – and shows one reason for doing so. What about lines?

Consider a line given in parametric form by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$ where

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} .$$

We need to find two equations

$$\mathbf{a}_1 \cdot (\mathbf{x} - \mathbf{x}_0) \quad \text{and} \quad \mathbf{a}_2 \cdot (\mathbf{x} - \mathbf{x}_0)$$

for two planes that both contain this line. The base point \mathbf{x}_0 is no problem; the base point \mathbf{x}_0 from the line will do. We just have to find \mathbf{a}_1 and \mathbf{a}_2 .

The vectors \mathbf{a}_1 and \mathbf{a}_2 are orthogonal to their respective planes, and since the lines and the planes have the same base point \mathbf{x}_0 in common, these planes will contain the line if and only if

$$\mathbf{a}_1 \cdot \mathbf{v} = 0 \quad \text{and} \quad \mathbf{a}_2 \cdot \mathbf{v} = 0 .$$

So all we need to do is to find a pair of linearly independent vectors \mathbf{a}_1 and \mathbf{a}_2 that are orthogonal to \mathbf{v} , the direction vector of the line. (If they are not linearly independent, the two planes would be the same, and their intersection would be plane, not a line).

We can use a trick from R^2 to do this. Recall that for any vector $\mathbf{c} = \begin{bmatrix} c \\ d \end{bmatrix}$ in R^2 , we get an orthogonal vector \mathbf{c}^\perp by swapping the entries and putting in a minus sign:

$$\text{If } \mathbf{c} = \begin{bmatrix} c \\ d \end{bmatrix} , \quad \text{then } \mathbf{c}^\perp = \begin{bmatrix} -d \\ c \end{bmatrix} ,$$

and as you easily check, $\mathbf{c} \cdot \mathbf{c}^\perp = 0$, so that \mathbf{c}^\perp is orthogonal to \mathbf{c} .

To apply this in R^3 , we begin with an example. Take $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Let \mathbf{a}_1 be given by

$$\mathbf{a}_1 = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} .$$

We have chosen to “zero out” the third component of \mathbf{a}_1 . This means that the third components do not enter into the dot product so

$$\mathbf{a}_1 \cdot \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and we are left with a dot product in R^2 . We know how to make this zero: Choose

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^\perp = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore, we have $\mathbf{a}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

To find \mathbf{a}_2 , we apply the same procedure, zeroing out a different entry this time. If we choose to zero out the first entry, so that $\mathbf{a}_2 = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$, we get $\mathbf{a}_2 \cdot \mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Hence

we take $a = b = 1$, and have $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Now that we have found \mathbf{a}_1 and \mathbf{a}_2 , we can write down the system of equations for the line in the standard algebraic form. Computing, we find

$$\mathbf{a}_1 \cdot \mathbf{x}_0 = 1 \quad \text{and} \quad \mathbf{a}_2 \cdot \mathbf{x}_0 = 3.$$

Also,

$$\mathbf{a}_1 \cdot \mathbf{x} = -x + y \quad \text{and} \quad \mathbf{a}_2 \cdot \mathbf{x} = y + z.$$

Hence the system of equations for the line is

$$\begin{aligned} -x + y &= 1 \\ y + z &= 3. \end{aligned} \tag{1.14}$$

There is an alternative method, based on the cross product, for finding a pair of vectors \mathbf{a}_1 and \mathbf{a}_2 that are orthogonal to \mathbf{v} , and linearly independent too. Here is how it goes:

First pick any vector \mathbf{w} that is not a multiple of \mathbf{v} . The standard choice is $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ since if \mathbf{v} is a multiple of this vector, everything is obvious anyway. Now choose

$$\mathbf{a}_1 = \mathbf{w} \times \mathbf{v}$$

and then

$$\mathbf{a}_2 = \mathbf{a}_1 \times \mathbf{v} .$$

This works because, by the properties of the cross product, $\mathbf{w} \times \mathbf{v}$ is a non-zero vector that is orthogonal to both \mathbf{w} and \mathbf{v} . In particular, it is orthogonal to \mathbf{v} , which is what we are after.

Again by the properties of the cross product $\mathbf{a}_1 \times \mathbf{v}$ is a non-zero vector that is orthogonal to both \mathbf{a}_1 and \mathbf{v} . In particular, it is orthogonal to \mathbf{v} , which is what we are after. So you

can also find \mathbf{a}_1 and \mathbf{a}_2 by computing two cross products. For example, with $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, we get $\mathbf{a}_1 = \mathbf{w} \times \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{a}_2 = \mathbf{a}_1 \times \mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$ With these choices

for \mathbf{a}_1 and \mathbf{a}_2 , we get the system of equations

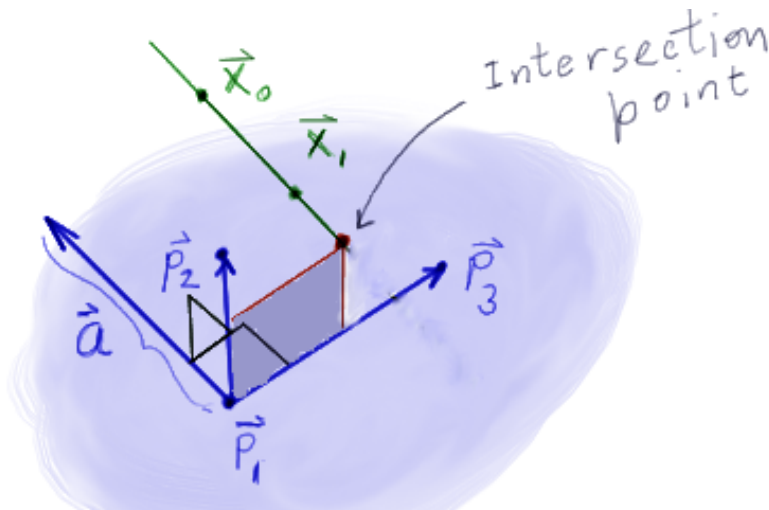
$$\begin{aligned} y + z &= 3 \\ -2x + y - z &= -1 . \end{aligned}$$

This is a different system than we found before, but both systems are *equivalent*; i.e., they have the same line as their solution set.

1.4 Lines intersecting planes

The material that we have just covered provides the means to answer a wide range of geometric problems. Here is an example:

Consider the plane that passes through three points, \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 , not all on the same line. Consider also a line ℓ that passes through two distinct points \mathbf{x}_0 and \mathbf{x}_1 . Unless the line happens to be parallel to the plane, it will intersect the plane in exactly one point. Here is an illustration:



We now ask: *how do we find the intersection point?*

Let's solve this problem in a concrete example. Let \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 be given by (1.10), and let \mathbf{x}_0 and \mathbf{x}_1 be given by

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} .$$

There are several ways to solve our problem; i.e., find the point of intersection

First solution: In this solution, we use the equation for the plane, and the parametric form of the line. We have already computed the equation of the plane above; it is given by (1.13), namely

$$x + y + z = 6 . \quad (1.15)$$

To parameterize the line, use $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_0$ as the direction vector. Calculating, we find

$$\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} . \text{ Therefore}$$

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v} = \begin{bmatrix} 1+t \\ 2+t \\ 1-t \end{bmatrix} . \quad (1.16)$$

Plugging $x(t)$, $y(t)$ and $z(t)$ into (1.15), we get

$$(1+t) + (2+t) + (1-t) = 6$$

or $t = 2$. Therefore, $\mathbf{x}(2)$ is in the plane. Computing,

$$\mathbf{x}(2) = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} .$$

This is the point we seek.

Second solution: In this solution, we use the equation for the plane, and the system of equations for the line. We have already worked out the equation of the plane in (1.13). Just above, we found the parametric form of the line. It is the one we used in our example of how to recover the system of equations of a line from the parametric form, hence we know that the system of equations for the line is given by (1.14).

The point we are seeking satisfies both (1.13) and (1.14), so it satisfies the combined system

$$\begin{aligned} x + y + z &= 6 \\ -x + y &= 1 \\ y + z &= 3 . \end{aligned} \quad (1.17)$$

The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array} \right]$$

which row reduces to

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Now back substitution gives us $z = -1$, then $y + z = 3$, so $y = 4$, and then $x + y + z = 6$, so $x = 3$. Again we find the point $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$.

The second solution was efficient once we found the system of equations for the line. To do this, we had to find the parameterization first. When a line is specified by giving two points on it, a parameterization is easy to write down, but finding the equations takes some work. This is also true when a plane is specified by giving three points on it. Hence there is some advantage to being able to work entirely with parameterizations. We do this in our third solution, which has other advantages as well.

Third solution: In this solution, we use the parametric descriptions of the plane and the line. As we have seen in (1.12), the parametric description of the plane is given by

$$\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$$

where

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} .$$

Let $\mathbf{z}(u)$ be a parameterization of the line. (The symbols \mathbf{x} and t are already taken, so we use \mathbf{z} and u to avoid confusion). As we have seen in (1.16), this is

$$\mathbf{z}(u) = \begin{bmatrix} 1 + u \\ 2 + u \\ 1 - u \end{bmatrix} .$$

We are looking for values of s , t , and u so that

$$\mathbf{x}(s, t) = \mathbf{z}(u) . \tag{1.18}$$

If we find them, we have a point that is on both the plane and the line.

Let's put the terms involving s and t on the left, and everything else on the right. This gives us the equation

$$s\mathbf{v}_1 + t\mathbf{v}_2 = \mathbf{z}(u) - \mathbf{x}_0 = \begin{bmatrix} u \\ u \\ -2 - u \end{bmatrix}. \quad (1.19)$$

To write this in matrix form, introduce

$$A = [\mathbf{v}_1, \mathbf{v}_2] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

Then since $A \begin{bmatrix} s \\ t \end{bmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2$, we have that (1.19) is equivalent to

$$A \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} u \\ u \\ -2 - u \end{bmatrix}. \quad (1.20)$$

Now let's solve this for $\begin{bmatrix} s \\ t \end{bmatrix}$. The corresponding augmented matrix is

$$\left[\begin{array}{cc|c} 2 & 0 & u \\ 0 & 1 & u \\ -2 & -1 & -2 - u \end{array} \right].$$

This row reduces to

$$\left[\begin{array}{cc|c} 2 & 0 & u \\ 0 & 1 & u \\ 0 & 0 & u - 2 \end{array} \right]. \quad (1.21)$$

We see that there is a pivot in the final column unless $u = 2$, and hence there is a solution only for $u = 2$. This means that the line crosses the plane at $\mathbf{z}(2)$, and we find

$$\mathbf{z}(2) = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}.$$

This is the point we seek.

We can now go further and find the s and t values. Here is why you might want to do this. Suppose we want to know whether the line hits the triangle with vertices at the points \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 . We will use the following fact, which is easily understood from a picture:

• The set of points lying on the triangle with vertices at any three non colinear points \mathbf{p}_1 , \mathbf{p}_2 , and \mathbf{p}_3 in R^3 are given by

$$\mathbf{p}_1 + s(\mathbf{p}_2 - \mathbf{p}_1) + t(\mathbf{p}_3 - \mathbf{p}_1)$$

where

$$s, t \geq 0 \quad \text{and} \quad s + t \leq 1 . \quad (1.22)$$

Since we have chosen $\mathbf{x}_0 = \mathbf{p}_1$, $\mathbf{v}_1 = (\mathbf{p}_2 - \mathbf{p}_1)$ and $\mathbf{v}_2 = (\mathbf{p}_3 - \mathbf{p}_1)$ in our parameterization, we have that the point of intersection, $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$, lies in the triangle if and only if its parameters s and t satisfy (1.22).

To check this we go on and solve $A \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$. We can use our previous work and just plug $u = 2$ into (1.21) getting $\begin{bmatrix} 2 & 0 & | & 2 \\ 0 & 1 & | & 2 \\ 0 & 0 & | & 0 \end{bmatrix}$. From here we see $t = 2$ and $s = 1$.

The conditions (1.22) are not satisfied, and the line passes through the plane outside the triangle.

1.5 Lines reflecting off planes

Let ℓ be a line with direction vector \mathbf{v} . We know how to find the point at which this line will hit any plane that is not parallel to the line. If the line describes the path of an incoming light ray, and the plane is a mirror, the light ray will be reflected, and another line will describe the path of the reflected ray. How can we find the second line corresponding to the reflected ray?

The first step is to find the point \mathbf{x}_0 where the incoming line crosses the plane. We know how to do this. Next, we need to find the direction vector of the outgoing line.

Here is how to do this: Let $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0)$ be the equation of the plane. Form the unit vector

$$\mathbf{u} = \frac{1}{|\mathbf{a}|} \mathbf{a} .$$

We now make the following claim, which will be applied with this unit vector \mathbf{u} , and the direction vector \mathbf{v} of the line:

• Let \mathbf{u} be any given unit vector in R^3 . Any vector \mathbf{v} in R^3 can be uniquely decomposed into a sum

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

where \mathbf{v}_{\parallel} is parallel to \mathbf{u} , and \mathbf{v}_{\perp} is orthogonal to \mathbf{u} . In fact, we have the formulas

$$\mathbf{v}_{\parallel} = (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \quad \text{and} \quad \mathbf{v}_{\perp} = \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} . \quad (1.23)$$

Proof: To justify this, notice that if \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} are defined by (1.23), then clearly $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$. Also, \mathbf{v}_{\parallel} is defined to be a multiple of \mathbf{u} , so certainly it is parallel to \mathbf{u} . Next, to check that \mathbf{v}_{\perp} is orthogonal to \mathbf{u} , compute its dot product with \mathbf{u} . Since \mathbf{u} is a unit vector, $\mathbf{u} \cdot \mathbf{u} = 1$, and so

$$\begin{aligned}\mathbf{v}_{\perp} \cdot \mathbf{u} &= (\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u}) \cdot \mathbf{u} \\ &= \mathbf{v} \cdot \mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \cdot \mathbf{u} \\ &= \mathbf{v} \cdot \mathbf{u} - (\mathbf{u} \cdot \mathbf{v})1 = 0.\end{aligned}$$

Finally, suppose that $\mathbf{v} = \tilde{\mathbf{v}}_{\parallel} + \tilde{\mathbf{v}}_{\perp}$ is another such decomposition. Then $\mathbf{v}_{\parallel} + \mathbf{v}_{\perp} = \tilde{\mathbf{v}}_{\parallel} + \tilde{\mathbf{v}}_{\perp}$ so that

$$\tilde{\mathbf{v}}_{\parallel} - \mathbf{v}_{\parallel} = \mathbf{v}_{\perp} - \tilde{\mathbf{v}}_{\perp}.$$

The vector on the left is parallel to \mathbf{u} , and the vector on the right is orthogonal to \mathbf{u} , so both vectors are orthogonal to themselves. But only the zero vector is orthogonal to itself, so

$$\tilde{\mathbf{v}}_{\parallel} - \mathbf{v}_{\parallel} = 0 \quad \text{and} \quad \mathbf{v}_{\perp} - \tilde{\mathbf{v}}_{\perp} = 0.$$

This shows that $\tilde{\mathbf{v}}_{\parallel} = \mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp} = \tilde{\mathbf{v}}_{\perp}$, so the formula (1.23) gives the unique decomposition of this type. ■

To apply this, let \mathbf{v} be the direction vector of the incoming line. Let \mathbf{u} be a unit vector that is orthogonal to the plane; i.e., one of $\pm \frac{1}{|\mathbf{a}|}\mathbf{a}$, and decompose \mathbf{v} as $\mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$. When the ray of light reflects off the plane, the component that is orthogonal to the plane changes sign. This is the component that is *parallel to* \mathbf{u} , so the new direction vector \mathbf{w} is

$$\mathbf{w} = \mathbf{v}_{\perp} - \mathbf{v}_{\parallel}.$$

From the formulas for \mathbf{v}_{\parallel} and \mathbf{v}_{\perp} , we have

$$\begin{aligned}\mathbf{w} &= \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{u} \\ &= \mathbf{v} - 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u}.\end{aligned}\tag{1.24}$$

Here is an example:

Consider the plane that passes through \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 where these vectors are given by (1.10). Consider the line that passes through \mathbf{x}_0 and \mathbf{x}_1 where

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Find the parametric description of the line obtained by reflecting this line off this plane.

In the previous section we have already computed that the point of intersection of the line and the plane is $\begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$, which we take as our base point \mathbf{x}_0 .

We have also found that the equation of the plane is $x + y + z = 6$, so the vector \mathbf{a} is $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and so we have the unit vector $\mathbf{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Finally, we found that the

direction vector of the line is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

This is all we need. To work out the reflected direction vector, we first work out $\mathbf{u} \cdot \mathbf{v} = 1/\sqrt{3}$. Therefore

$$\mathbf{w} = \mathbf{v} - 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}.$$

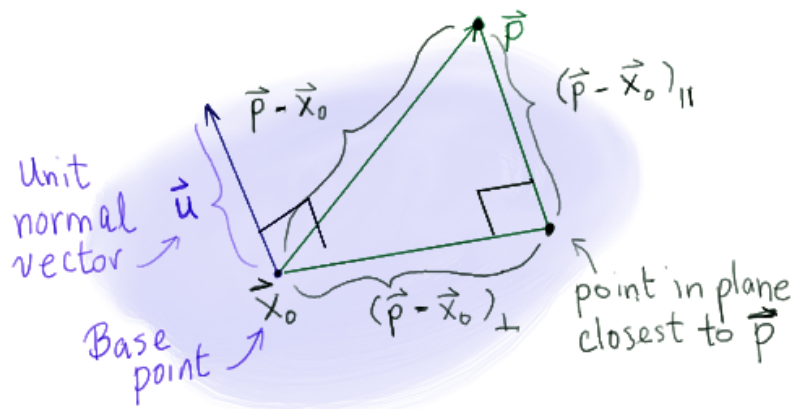
The reflected line then is $\mathbf{x}_0 + u\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} + \frac{u}{3} \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$.

1.6 Distance problems

The orthogonal decomposition (1.24) is also the key to distance problems. For example, let \mathbf{p} be any point in R^3 , and consider any plane given by an equation $\mathbf{a} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$. Then the vector $\mathbf{p} - \mathbf{x}_0$ runs from the base point \mathbf{x}_0 in the plane to \mathbf{p} , but in general its length is greater than the distance between \mathbf{p} and the plane. To find the distance, let \mathbf{u} be either unit vector orthogonal to the plane, and decompose

$$(\mathbf{p} - \mathbf{x}_0) = (\mathbf{p} - \mathbf{x}_0)_{\parallel} + (\mathbf{p} - \mathbf{x}_0)_{\perp}.$$

Here is an illustration:



Only the parallel component is relevant for computing the distance from the plane to \mathbf{p} , as you see above. Hence, the distance equals the length of parallel component $|(\mathbf{p} - \mathbf{x}_0)_\parallel|$.

Let's compute the distance from $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to the plane given by $x + 2y + z = 1$. We

get a particular solution of this equation by taking $y = z = 0$ and then $x = 1$, so we can

take $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then $\mathbf{p} - \mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

The normal vector to the plane is $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, so we have the unit vector $\mathbf{u} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Then, using (1.24), we compute $(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{u} = 3/\sqrt{6}$, so that $(\mathbf{p} - \mathbf{x}_0)_\parallel = \frac{3}{\sqrt{6}}\mathbf{u} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

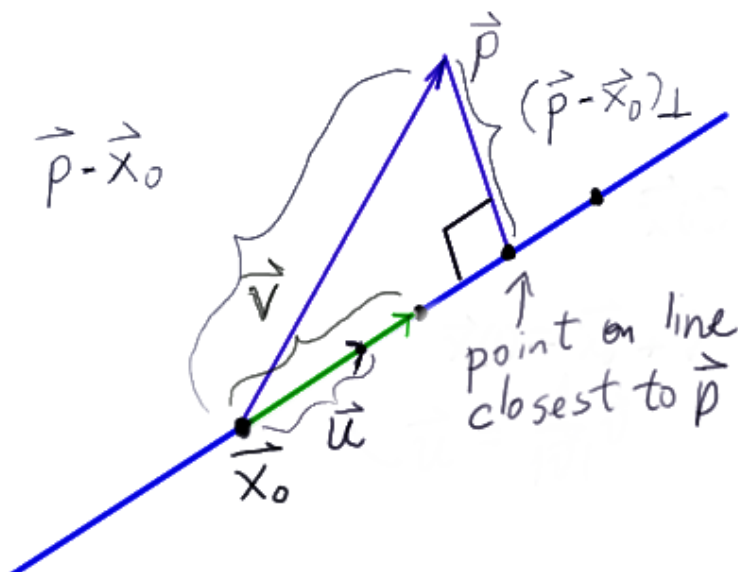
Hence, $|(\mathbf{p} - \mathbf{x}_0)_\parallel| = 3/\sqrt{6}$, and this is the distance from \mathbf{p} to the plane.

Computing the distance from a point to a line is similar: Let ℓ be the line given in parametric form by $\mathbf{x}_0 + t\mathbf{v}$, and let \mathbf{p} be any given point in R^3 . Then the vector $\mathbf{p} - \mathbf{x}_0$ runs from the base point \mathbf{x}_0 on the line to \mathbf{p} , but in general its length is greater than the distance between \mathbf{p} and the line. To find the distance, let \mathbf{u} be the unit vector parallel to \mathbf{v} , i.e.,

$$\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v},$$

and decompose

$$(\mathbf{p} - \mathbf{x}_0) = (\mathbf{p} - \mathbf{x}_0)_\parallel + (\mathbf{p} - \mathbf{x}_0)_\perp.$$



This time, the parallel component corresponds to motion along the line, which has nothing to do with the distance from the line, and it is the length of the orthogonal component, $|(\mathbf{p} - \mathbf{x}_0)_\perp|$.

For example, consider the line given by $\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v}$ where

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Let $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then we have $\mathbf{p} - \mathbf{x}_0 = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$, and $\mathbf{u} = \frac{1}{|\mathbf{v}|}\mathbf{v} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. We compute $(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{u} = 1/\sqrt{3}$. Hence, from the formula (1.23) we have

$$(\mathbf{p} - \mathbf{x}_0)_\perp = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

The length of this vector is $\sqrt{14/3}$, and so that is the distance from \mathbf{p} to ℓ .

Problems

1.1 Consider the plane passing through the three points

$$\mathbf{p}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

and the line passing through

$$\mathbf{z}_0 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{z}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

- Find a parametric representation $\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.
- Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.
- Find an equation for the plane.
- Find a system of equations for the line.
- Find the points, if any, where the line intersects the plane.
- Does the line pass through the triangle with vertices \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 ? Justify your answer with a computation.
- Find the distance from \mathbf{p}_1 to the line.
- Find the distance from \mathbf{z}_0 to the plane.

1.2 Consider the plane passing through the three points

$$\mathbf{p}_1 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

and the line passing through

$$\mathbf{z}_0 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{z}_1 = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

- (a) Find a parametric representation $\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.
- (b) Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.
- (c) Find an equation for the plane.
- (d) Find a system of equations for the line.
- (e) Find the points, if any, where the line intersects the plane.
- (f) Does the line pass through the triangle with vertices \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 ? Justify your answer with a computation.
- (g) Find the distance from \mathbf{p}_1 to the line.
- (h) Find the distance from \mathbf{z}_0 to the plane.

1.3 Consider the plane passing through the three points

$$\mathbf{p}_1 = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} \quad \mathbf{p}_2 = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_3 = \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix}$$

and the line passing through

$$\mathbf{z}_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{z}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

- (a) Find a parametric representation $\mathbf{x}(s, t) = \mathbf{x}_0 + s\mathbf{v}_1 + t\mathbf{v}_2$ for the plane.
- (b) Find a parametric representation $\mathbf{z}(u) = \mathbf{z}_0 + u\mathbf{w}$ for the line.
- (c) Find an equation for the plane.
- (d) Find a system of equations for the line.
- (e) Find the points, if any, where the line intersects the plane.
- (f) Does the line pass through the triangle with vertices \mathbf{p}_1 , \mathbf{p}_2 and \mathbf{p}_3 ? Justify your answer with a computation.
- (g) Find the distance from \mathbf{p}_1 to the line.
- (h) Find the distance from \mathbf{z}_0 to the plane.

1.4 Consider the plane given by

$$2x - y + 3z = 4 .$$

Let $\mathbf{p} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$. What is the distance from \mathbf{p} to the plane?

1.5 Consider the plane given by

$$x - 3y + z = 2 .$$

Let $\mathbf{p} = \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$. What is the distance from \mathbf{p} to the plane?

1.6 Consider the line given by

$$\begin{aligned} 2x - y + 3z &= 4 \\ x + y + z &= 2 . \end{aligned}$$

Let $\mathbf{p} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. What is the distance from \mathbf{p} to the line?

1.7 Consider the line given by

$$\begin{aligned}x - 3y + z &= 2 \\ 2y + z &= 3 .\end{aligned}$$

Let $\mathbf{p} = \begin{bmatrix} -2 \\ -5 \\ 1 \end{bmatrix}$. What is the distance from \mathbf{p} to the line?

1.8 Consider the line ℓ given by

$$\begin{aligned}2x - y + 3z &= 4 \\ x + y + z &= 2 .\end{aligned}$$

Find a parametric representation of the line obtained by reflecting this line through the plane

$$x + 3y - z = 1 .$$

1.9 Consider the line ℓ given by

$$\begin{aligned}x - 3y + z &= 2 \\ 2y + z &= 3 .\end{aligned}$$

Find a parametric representation of the line obtained by reflecting this line through the plane

$$x + 2y - z = 1 .$$

1.10 Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$. Find two linearly independent vectors \mathbf{a}_1 and \mathbf{a}_2 that are orthogonal to \mathbf{v} .

1.11 Consider the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Find two linearly independent vectors \mathbf{a}_1 and \mathbf{a}_2 that are orthogonal to \mathbf{v} .

Section 2: The distance between two lines in \mathbb{R}^3

2.1: An algebraic point of view

Consider two lines in \mathbb{R}^3 given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$. The distance between the points $\mathbf{x}_1(s)$ and $\mathbf{x}_2(t)$ is $|\mathbf{x}_1(s) - \mathbf{x}_2(t)|$. As s and t vary, so does the distance between the points. The *distance between the two lines* is, by definition, the minimum value of $|\mathbf{x}_1(s) - \mathbf{x}_2(t)|$ as s and t range independently over the real numbers. Computing this distance is a least squares problem, as we now explain.

First of all, we will seek the values of s and t that minimize the squared distance $|\mathbf{x}_1(s) - \mathbf{x}_2(t)|^2$. If we can find s_0 and t_0 so that

$$|\mathbf{x}_1(s_0) - \mathbf{x}_2(t_0)|^2 \leq |\mathbf{x}_1(s) - \mathbf{x}_2(t)|^2 \quad \text{for all } s, t, \quad (2.1)$$

then taking square roots of both sides we have

$$|\mathbf{x}_1(s_0) - \mathbf{x}_2(t_0)| \leq |\mathbf{x}_1(s) - \mathbf{x}_2(t)| \quad \text{for all } s, t, \quad (2.2)$$

since the square root function is monotone increasing. Hence $|\mathbf{x}_1(s_0) - \mathbf{x}_2(t_0)|$ will be the distance between the lines.

Now note that

$$\mathbf{x}_1(s) - \mathbf{x}_2(t) = s\mathbf{v}_1 - t\mathbf{v}_2 - (\mathbf{x}_2 - \mathbf{x}_1).$$

If we define A to be the matrix $A = [\mathbf{v}_1, -\mathbf{v}_2]$, and $\mathbf{b} = \mathbf{x}_2 - \mathbf{x}_1$, and finally $\mathbf{s} = \begin{bmatrix} s \\ t \end{bmatrix}$, we can rewrite this as

$$\mathbf{x}_1(s) - \mathbf{x}_2(t) = A\mathbf{s} - \mathbf{b}.$$

Therefore,

$$|\mathbf{x}_1(s) - \mathbf{x}_2(t)|^2 = |A\mathbf{s} - \mathbf{b}|^2,$$

and so if $\mathbf{s}_0 = \begin{bmatrix} s_0 \\ t_0 \end{bmatrix}$ is the least squares solution of

$$A\mathbf{s} = \mathbf{b},$$

then (2.2) holds, and the distance between the two lines is $|A\mathbf{s}_0 - \mathbf{b}|$.

Example 1 (The distance between two lines) Let \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{v}_1 and \mathbf{v}_2 be given by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}.$$

The normal equations are

$$A^t A\mathbf{s} = A^t \mathbf{b}.$$

We compute $A^t A = \begin{bmatrix} 9 & 3 \\ 3 & 2 \end{bmatrix}$, so that $(A^t A)^{-1} = \frac{1}{9} \begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix}$. We then compute $A^t \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, so that

$$\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 2 & -3 \\ -3 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} .$$

Hence $s_0 = 1/3$, and $t_0 = 0$.

We now compute

$$\mathbf{x}_1(1/3) = \frac{1}{3} \begin{bmatrix} 5 \\ 5 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(0) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} .$$

Hence $\mathbf{x}_1(1/3) - \mathbf{x}_2(0) = \frac{1}{3} \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$ and so

$$|\mathbf{x}_1(1/3) - \mathbf{x}_2(0)| = \frac{\sqrt{36}}{3} = 2 .$$

This is the distance between the two lines; for no values of s and t is $|\mathbf{x}_1(s) - \mathbf{x}_2(t)|$ any smaller than this, the value at $s = 1/3$ and $t = 0$.

2.2: A differential calculus point of view

Define a function f of two variables s and t by

$$f(s, t) = |\mathbf{x}_1(s) - \mathbf{x}_2(t)|^2 .$$

This represents the square of the distance between $\mathbf{x}_1(s)$ and $\mathbf{x}_2(t)$. Finding values of s and t that make $f(s, t)$ as small as possible amounts to computing the square of the distance between the two lines, and hence doing so determines the distance.

Example 2 (A two variable distance function) Let \mathbf{x}_1 , \mathbf{x}_2 , \mathbf{v}_1 and \mathbf{v}_2 be given as in Example 1. Then

$$\begin{aligned} f(s, t) &= (2s + t - 2)^2 + (2s)^2 + (s + t + 1)^2 \\ &= 9s^2 + 2t^2 + 6st - 6s - 2t + 5 . \end{aligned}$$

In the last subsection, we saw how to compute values s_0 and t_0 so that (2.2) holds, or, what is the same, so that

$$f(s_0, t_0) \leq f(s, t) \quad \text{for all} \quad s, t . \quad (2.3)$$

Now, fix $s = s_0$, and define a function $g(t)$ by

$$g(t) = f(s_0, t) .$$

From the definition and (2.3),

$$g(t_0) \leq g(t)$$

for all t . This means that g has a minimum at $t = t_0$, and from the calculus of functions of a single variable, we know that this means that the derivative of g is zero at $t = t_0$. That is:

$$g'(t_0) = 0 . \quad (2.4)$$

In the same way, fix $t = t_0$, and define a function $h(s)$ by

$$h(s) = f(s, t_0) .$$

From the definition and (2.3),

$$h(s_0) \leq h(s)$$

for all s . Hence h has a minimum at $s = s_0$, and so

$$h'(s_0) = 0 . \tag{2.5}$$

The two equations (2.4) and (2.5) can be solved to determine the two unknowns s_0 and t_0 . Notice that both equations involve both s_0 and t_0 through the definitions of g and h , so that (2.4) and (2.5) forms a *system of equations*. Notice the strategy:

- *We are going to determine the minimum of a function of two variables by considering one variable at a time, thus placing us on the familiar ground of single variable calculus.*

Example 3 Let $f(s, t)$ be the function determined in Example 2:

$$f(s, t) = 9s^2 + 2t^2 + 6st - 6s - 2t + 5 .$$

Then

$$g(t) = 2t^2 + [6s_0 - 2]t + [9s_0^2 - 6s_0 + 5]$$

where we have group the terms by powers of t . Differentiating with respect to the variable t , $g'(t) = 4t + [6s_0 - 2]$, and so (2.4) becomes

$$4t_0 + 6s_0 = 2 . \tag{2.6}$$

Likewise,

$$h(s) = 9s^2 + [6t_0 - 6]s + [2t_0^2 - 2t_0 + 5] .$$

Differentiating with respect to the variable s , $h'(s) = 18s + [6t_0 - 6]$, and so (2.4) becomes

$$18s_0 + 6t_0 = 6 . \tag{2.7}$$

Dividing through by 2 and 6 respectively, (2.6) and (2.7) give us the system

$$\begin{aligned} 3s_0 + 2t_0 &= 1 \\ 3s_0 + t_0 &= 1 \end{aligned}$$

Subtracting the second equation from the first, we find $t_0 = 0$, and then we see $s_0 = 1/3$. Hence the minimum occurs at $\begin{bmatrix} s_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix}$, as we found before.

2.3: Comparing the two points of view

For computing the distance between to lines, the purely algebraic approach worked out in Example 1 is very efficient. However, the approach worked out in Example 3, in which we apply the methods of differential calculus *one variable at a time* has a significant advantage: It is easy to generalize this method to seek minimum and maximum values of a rich variety of functions of several variables. If a function involves third or higher powers

of its variables, let alone transcendental functions of them, one cannot hope to write the minimization problem as a least squares problem. But one can still hope to apply the methods of differential calculus one variable at a time.

Does this mean that the algebraic point of view is superseded by the differential calculus point of view? Not at all! As we shall see, the most powerful way to solve many problems is use both points of view together. The next chapter introduces certain vectors and matrices, namely gradients, Hessians and Jacobians, that are defined in terms of derivatives. Algebraic analysis of these objects will be the path of progress in much of our work. Our first order of business in the next chapter though is to see how far we can go with the *one variable at a time* strategy.*

Problems

1. Consider two lines in \mathbb{R}^3 given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} .$$

Compute the distance between these two lines.

2. Consider two lines in \mathbb{R}^3 given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} .$$

Compute the distance between these two lines.

3. Consider two lines in \mathbb{R}^3 given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} .$$

Compute the distance between these two lines.

4. Consider two lines in \mathbb{R}^3 given parametrically by $\mathbf{x}_1(s) = \mathbf{x}_1 + s\mathbf{v}_1$ and $\mathbf{x}_2(t) = \mathbf{x}_2 + t\mathbf{v}_2$ where

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ -5 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix} .$$

Compute the distance between these two lines.

* This is a good place to emphasize that to master this subject, one must approach it with the goal of learning *problem solving strategies* and not just formulas. The formulas *are* important, but still they are only the foot soldiers needed to advance the strategies, and there are many ways to employ them. Choosing one that succeeds is a strategic decision, and unless one develops strategic vision, the largest army of formulas will be of no avail.

Section 3: Functions from \mathbb{R} to \mathbb{R}^n

3.1 Curves in \mathbb{R}^n

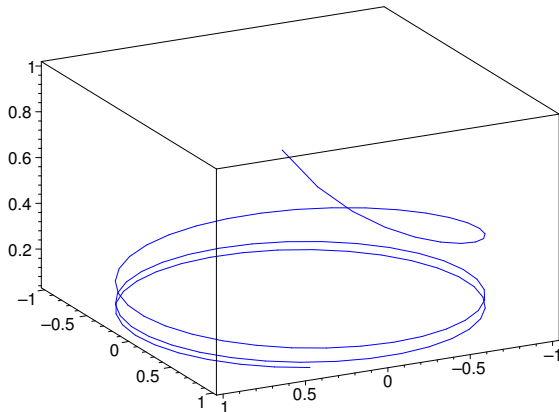
In many ways, the simplest sort of multivariable functions are functions from \mathbb{R} to \mathbb{R}^n for some $n \geq 1$. These are functions that have one input variable (one independent variable), and n output variables (n dependent variables).

If $n = 2$ or if $n = 3$, we can think of the output variables as giving the coordinates of a point moving in \mathbb{R}^2 or \mathbb{R}^3 , and we can think of the input variables as the time, so that the function gives us the location of a moving point at time t . Such functions describe curves in \mathbb{R}^2 or \mathbb{R}^3 . For this reason, it is natural to use \mathbf{x} for the dependent variables, and t for the independent variable, and to write the functions as $\mathbf{x}(t)$. Such functions are also called *vector valued functions of a real variable*, for obvious reasons.

For example, consider a function $\mathbf{x}(t)$ of the real variable t with values in \mathbb{R}^n . For example, let's consider $n = 3$, and

$$\mathbf{x}(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ 1/t \end{bmatrix}. \quad (1.1)$$

Here is a three dimensional plot of the curve traced out by $\mathbf{x}(t)$ as t varies from $t = 1$ to $t = 20$:



Such vector valued functions arise whenever we need to describe the position of a particle as a function of time. But more generally, we might have any sort of system that is described by n parameters. These could be, for example, the voltages across n points in an electric circuit. We can arrange this data into a vector, and if the data is varying with time, as is often the case in applications, we then have a time dependent vector $\mathbf{x}(t)$ in \mathbb{R}^n .

When quantities are varying in time, it is often useful to consider their rates of change; i.e., derivatives.

Definition (Derivatives of Vector Valued Functions) Let $\mathbf{x}(t)$ be a vector valued function of the variable t . We say that $\mathbf{x}(t)$ is *differentiable at $t = t_0$* with derivative $\mathbf{x}'(t_0)$ in case

$$\lim_{h \rightarrow 0} \frac{1}{h} (\mathbf{x}(t_0 + h) - \mathbf{x}(t_0)) = \mathbf{x}'(t_0)$$

in the sense that this limit exists for each of the n entries separately. A vector valued function is *differentiable* in some interval (a, b) if it is differentiable for each t_0 in (a, b) .

There is nothing really new going on here. To compute the derivative of $\mathbf{x}(t)$, you just differentiate it entry by entry in the usual way.

Indeed, consider a t dependent vector $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ in \mathbb{R}^2 . Then, by the rules for vector subtraction and scalar multiplication,

$$\begin{aligned} \frac{1}{h} (\mathbf{x}(t+h) - \mathbf{x}(t)) &= \frac{1}{h} \left(\begin{bmatrix} x(t+h) \\ y(t+h) \end{bmatrix} - \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \right) \\ &= \begin{bmatrix} (x(t+h) - x(t))/h \\ (y(t+h) - y(t))/h \end{bmatrix} \end{aligned}$$

Now taking the limits on the right, entry by entry, we see that $\mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ provided $x(t)$ and $y(t)$ are both differentiable. The same reduction to single variable differentiation clearly extends to any number of entries.

Example 1 (Computing the derivative of a vector valued function of t) Let $\mathbf{x}(t)$ be given by (1.1). Then for any $t \neq 0$,

$$\mathbf{x}'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ -1/t^2 \end{bmatrix} .$$

Because we just differentiate vectors entry by entry without mixing the entries up in any way, familiar rules for differentiating numerically valued functions hold for vector valued functions as well. In particular, the derivative of a sum is still the sum of the derivatives, etc.:

$$(\mathbf{x}(t) + \mathbf{y}(t))' = \mathbf{x}'(t) + \mathbf{y}'(t) \quad (1.2).$$

Things are only slightly more complicated with the product rule because now we have several types of products to consider. Here is an example that we shall need soon: a “product rule” for the dot product.

Theorem 1 (Differentiating Dot Products) *Suppose that $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are differentiable vector valued functions for t in (a, b) with values in \mathbb{R}^n . Then $\mathbf{v}(t) \cdot \mathbf{w}(t)$ is differentiable for t in (a, b) , and*

$$\frac{d}{dt} \mathbf{v}(t) \cdot \mathbf{w}(t) = \mathbf{v}'(t) \cdot \mathbf{w}(t) + \mathbf{v}(t) \cdot \mathbf{w}'(t) . \quad (1.3)$$

Proof: For each i , we have by the usual product rule

$$\frac{d}{dt} v_i(t) w_i(t) = v_i'(t) w_i(t) + v_i(t) w_i'(t) .$$

Summing on i now gives us (1.3). ■

3.2 Velocity and acceleration

Let $\mathbf{x}(t)$ represent the position of a point particle at time t . The first time derivative of position is called the *velocity*. This gives you the rate of change of the position. The second time derivative of the position is called the *acceleration*. It gives you the rate of change of the velocity vector.

Consider $x(t)$, the x coordinate of $\mathbf{x}(t)$. This is a garden variety real valued function of a single real variable. By Taylor's Theorem, given any t_0 ,

$$x(t) = x(t_0) + x'(t_0)(t - t_0) + \frac{1}{2}x''(t_0)(t - t_0)^2 + \mathcal{O}((t - t_0)^3). \quad (1.4)$$

Doing the same for the other coordinates, and combining the results in vector form, we have

$$\mathbf{x}'(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{x}'(t_0) + \frac{(t - t_0)^2}{2}\mathbf{x}''(t_0) + \mathcal{O}((t - t_0)^3). \quad (1.5)$$

In the vector form, we have written the scalar multiples in front of the vectors, as we usually do.

You see here what the velocity tells you: If $h = t - t_0$ is small, then where you will be at time t is in leading order of approximation $\mathbf{x}(t_0) + h\mathbf{v}(t_0)$. The velocity tells you what increment in your position corresponds to a small increment h in time, namely $h\mathbf{v}(t_0)$.

The straight line function $\mathbf{z}(t)$ given by

$$\mathbf{z}(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{v}(t_0)$$

is called the *tangent line to $\mathbf{x}(t)$ at t_0* . It gives the best linear approximation to the curve $\mathbf{x}(t)$ at t_0 :

$$\mathbf{x}'(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{v}(t_0) + \mathcal{O}((t - t_0)^2). \quad (1.6)$$

Example 2 (Taylor approximation for a curve) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = \begin{bmatrix} 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$. Then for all $t > 0$,

$$\mathbf{x}'(t) = \begin{bmatrix} 1 \\ 2^{1/2}t^{1/2} \end{bmatrix}$$

and

$$\mathbf{x}''(t) = \begin{bmatrix} 0 \\ 2^{-1/2}t^{-1/2} \\ 1 \end{bmatrix}.$$

Taking $t_0 = 1$,

$$\mathbf{x}(1) = \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} \quad \mathbf{x}'(1) = \begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}''(1) = \begin{bmatrix} 0 \\ 2^{-1/2} \\ 1 \end{bmatrix}.$$

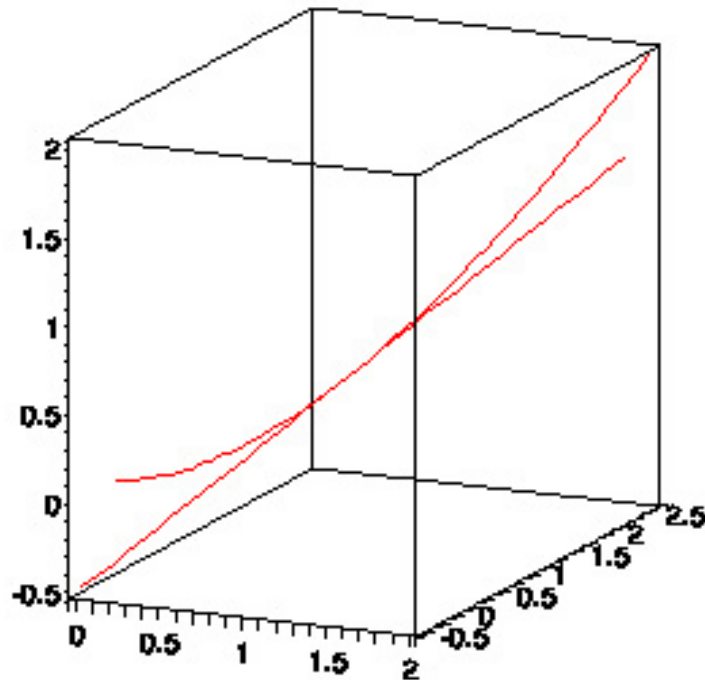
Therefore, when $t \approx 1$,

$$\mathbf{x}(t) \approx \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} + (t - 1) \begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} + \frac{(t - 1)^2}{2} \begin{bmatrix} 0 \\ 2^{-1/2} \\ 1 \end{bmatrix}.$$

We get the tangent line by just keeping the linear term in this approximation. Hence the tangent line at $t = 1$ is given in parametric form by

$$\mathbf{z}(t) = \begin{bmatrix} 1 \\ 2^{3/2}/3 \\ 1/2 \end{bmatrix} + (t-1) \begin{bmatrix} 1 \\ 2^{1/2} \\ 1 \end{bmatrix} .$$

Here is a plot showing the curve for $0 \leq t \leq 2$, together with the tangent line at $t = 1$. Observe the tangency.



Notice that for a vector valued function \mathbf{z} of the form $\mathbf{z} = \mathbf{x}_0 + t\mathbf{v}$, $\mathbf{z}''(t) = 0$ for all t . The acceleration is zero for such a linear path. This is characteristic of linear paths. Indeed, if $\mathbf{x}''(t) = 0$, then

$$\mathbf{x}(t) = \mathbf{x}(t_0) + (t - t_0)\mathbf{x}'(t_0) .$$

Taylor expansion stops exactly after the linear term. The error term in (1.6) depends on the second derivatives, but if $\mathbf{x}'' = 0$, there is no error term.

In general though, the velocity is not constant, and (1.5) gives a better approximation than (1.4). One could go on and consider higher derivatives. However, it is rarely useful to do so.* The reason is that Newton's second law relates the acceleration of point particle to the forces acting on it. Once the physical nature of the force is determined, the acceleration is determined, and the motion can be determined.

The acceleration and velocity are both vector quantities. Henceforth, we will write $\mathbf{v}(t)$ in place of $\mathbf{x}'(t)$ to denote the velocity, and write $\mathbf{a}(t)$ in place of $\mathbf{x}''(t)$ to denote the acceleration.

* The third derivative \mathbf{x}''' does have a name; it is called the *jerk*, at least in some references.

The magnitude of the velocity vector is called the *speed*. We denote it by $v(t)$. That is,

$$v(t) = |\mathbf{v}(t)| .$$

As you see from (1.6), when h is small,

$$v(t) \approx \frac{|\mathbf{x}(t+h) - \mathbf{x}(t)|}{h}$$

so that $v(t)$ is measuring the rate of increase in distance travelled. This is what we mean by speed.

Assuming that $v(t) \neq 0$, we can define a unit vector valued function $\mathbf{T}(t)$ by

$$\mathbf{T}(t) = \frac{1}{v(t)} \mathbf{v}(t) . \quad (1.7)$$

Then clearly

$$\mathbf{v}(t) = v(t) \mathbf{T}(t) . \quad (1.8)$$

The vector $\mathbf{T}(t)$ is called the *unit tangent vector* at time t . It gives the instantaneous direction of motion at time t . This factorization of the velocity vector into a unit vector giving the direction of motion, and a scalar multiple giving the speed of motion in that direction provides a very good way to think about the motion of point particles, as we shall explain.

Example 3 (Speed and the unit tangent vector) Let $\mathbf{x}(t)$ be given by $\mathbf{x}(t) = \begin{bmatrix} 2^{3/2}t^{3/2}/3 \\ t^2/2 \end{bmatrix}$ as in

the previous example. Then, as we have seen, for all $t > 0$, $\mathbf{x}'(t) = \begin{bmatrix} 2^{1/2}t^{1/2} \\ t \end{bmatrix}$. We then easily compute that

$$v(t) = \sqrt{1 + 2t + t^2} = 1 + t ,$$

and so

$$\mathbf{T}(t) = \frac{1}{1+t} \begin{bmatrix} 2^{1/2}t^{1/2} \\ t \end{bmatrix} .$$

Problems

1 Let $\mathbf{x}(t) = r \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ where $r > 0$. This is a parameterization of the unit circle.

(a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.

(b) Compute $v(t)$ and $\mathbf{T}(t)$.

(c) Find the tangent line to this curve at $t = \pi/4$. Also find the tangent line at $t = \pi/2$. Where do these lines intersect?

2 Let $\mathbf{y}(t) = \begin{bmatrix} t+1 \\ t^2 \end{bmatrix}$. This is a parameterization of the parabola $y = (x-1)^2$. (The reason we are using \mathbf{y} instead of \mathbf{x} will become clear in problem 5).

- (a) Compute $\mathbf{v}(t) = \mathbf{y}'(t)$ and $\mathbf{a}(t) = \mathbf{y}''(t)$.
- (b) Compute $v(t)$ and $\mathbf{T}(t)$.
- (c) Find the tangent line to this curve at $t = 1$.

3 Let $\mathbf{x}(t) = \begin{bmatrix} t \\ 2\sqrt{t} \\ 1/t \end{bmatrix}$.

- (a) Compute $\mathbf{v}(t)$ and $\mathbf{a}(t)$.
- (b) Compute $v(t)$ and $\mathbf{T}(t)$.
- (c) Find the tangent line to this curve at $t = 1$. Also find the tangent line at $t = \pi/2$. Do these lines intersect?

4 Let $\mathbf{y}(t) = \begin{bmatrix} \sqrt{t} \\ 2/\sqrt{t} \\ t \end{bmatrix}$. (The reason we are using \mathbf{y} instead of \mathbf{x} will become clear in problem 5).

- (a) Compute $\mathbf{v}(t) = \mathbf{y}'(t)$ and $\mathbf{a}(t) = \mathbf{y}''(t)$.
- (b) Compute $v(t)$ and $\mathbf{T}(t)$.
- (c) Find the tangent line to this curve at $t = 1$.

5 Let $\mathbf{x}(t)$ be the curve from problem 1, and let $\mathbf{y}(t)$ be the curve from problem 2.

- (a) Compute $f(t) = \mathbf{x}(t) \cdot \mathbf{y}(t)$, and then compute $f'(t)$ by direct differentiation.
- (b) Compute $\mathbf{x}'(t) \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \mathbf{y}'(t)$ using your results from problems 1 and 2, and then check that indeed $(\mathbf{x}(t) \cdot \mathbf{y}(t))' = \mathbf{x}'(t) \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \mathbf{y}'(t)$.

6 Let $\mathbf{x}(t)$ be the curve from problem 3, and let $\mathbf{y}(t)$ be the curve from problem 4.

- (a) Compute $f(t) = \mathbf{x}(t) \cdot \mathbf{y}(t)$, and then compute $f'(t)$ by direct differentiation.
- (b) Compute $\mathbf{x}'(t) \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \mathbf{y}'(t)$ using your results from problems 3 and 4, and then check that indeed $(\mathbf{x}(t) \cdot \mathbf{y}(t))' = \mathbf{x}'(t) \cdot \mathbf{y}(t) + \mathbf{x}(t) \cdot \mathbf{y}'(t)$.