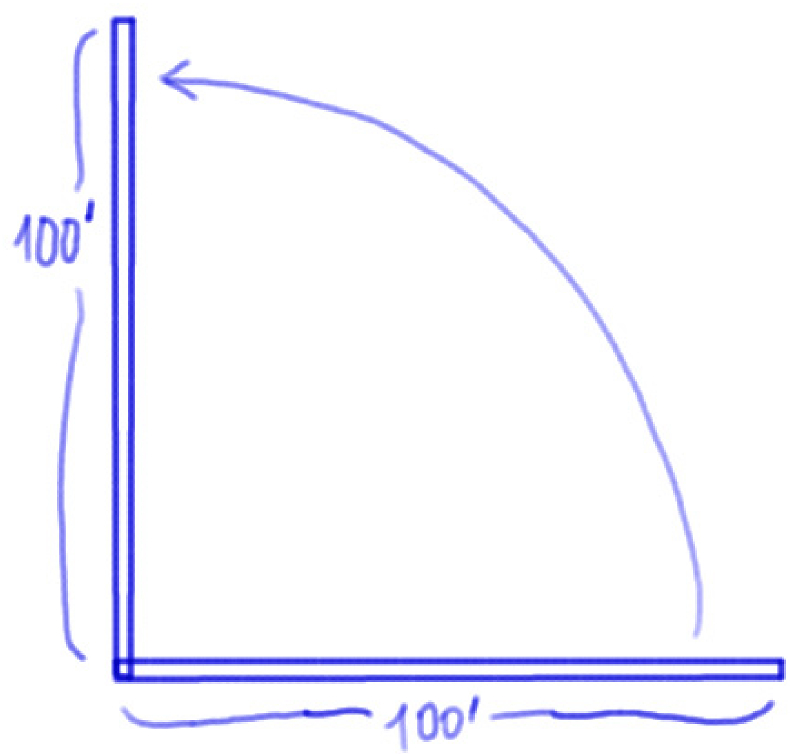
Section 1: Integrals in $\mathbb{R}^2$

1.1: Integrals on one variable

This chapter is focused on the problem of integrating functions of several variables. There are many ways to think about what it means to integrate a function of one variable, and not all of them are useful starting points for the transition to several variables. For example, many students think of integration as the procedure that "undoes differentiation". Indeed, in practice, one computes integrals by finding antiderivatives. But suppose we have a function $f(x, y)$ of two variables. What could it possibly mean to find an "antiderivative" of $f(x, y)$? We have come to understand the gradient as the derivative in two variables, but that is a vector quantity, and so it would make no sense to seek an "antigradient" of $f(x, y)$.

Therefore, we begin with a problem in one variable. Our aim is to explain *what integrals are* from a point of view that facilitates the transition to several variables. Consider the following problem:

• *How much work is required to raise a 100 foot flagpole that weighs one pound per foot from horizontal to vertical?*



Recall that *work* is product of force and the distance traveled in moving against the direction of that force. (Force is a vector quantity, so it has a direction). In the case at hand, the force is gravity. The direction is "straight down", so we are only concerned with vertical displacement in this case.

If we lift a one pound weight one foot, we do one foot–pound of work. If we lift a one pound weight ten feet, we do ten foot–pounds of work. This is all simple multiplication.

The flagpole problem, however, requires calculus because different parts of the flagpole get raised different amounts. Near the base, there is not much raising going on at all, while the parts of the flagpole near the top are raised nearly 100 feet. We cannot simply use the formula

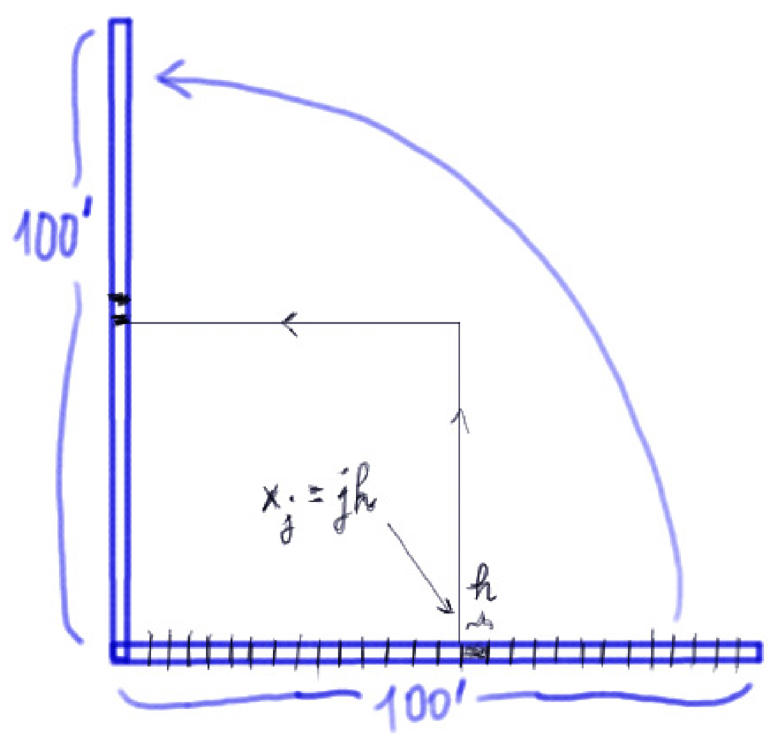
$$\text{work} = \text{weight} \ \times \ \text{height raised} \tag{1.1}$$

because there is no one value for "height lifted" that is valid for the whole flagpole. If we were going to keep it horizontal, but just lift it up 100 feet, we could use the formula, and then the total work would be

$$(100 \text{ pounds}) \ \times \ \ (100 \text{ feet}) = (10,000 \text{ footpounds}) \ .$$

The way to carry out the computation using calculus is to first chop the flagpole to small bits – in your mind only; do not ruin the flagpole. Pick a small distance step $h > 0$, and slice the flagpole perpendicular to its axis into little blocks that are h feet long. Now "raise " the flagpole by stacking the blocks in the right order. We are going to add up the amount of work we do lifting each block into place to get the total work done lifting the flagpole into place.

The jth block from the base will have to be raised to a height of jh feet. Not everything in the block gets raised by *exactly* this amount, but if h is small compared with jh, this will be a small difference percentagewise. Hence, up to a small percentagewise error, we can use the formula (1.1). Since the flagpole weighs 1 pound per foot, the weight of the block is h pounds.



Hence the work done in raising the jth block is

$$(h \text{ pounds}) \ \times \ \ (jh \text{ feet}) = (jh^2 \text{ footpounds}) \ .$$

Now the key point is that work is an *extensive*, or in other words, *additive*, quantity. Hence, letting N be the total number of blocks, which is $100/h$, we have that the total work is

$$\sum_{j=1}^{N} jh^2 \ .$$

Letting x_j denote jh, and letting Δx denote h, this becomes

$$\sum_{j=1}^{N} x_j \Delta x \ ,$$

and you recognize this as a Riemann sum for the integral

$$\int_0^{100} x\delta x = \left.\frac{x^2}{2}\right|_0^{100} = 5,000 \ .$$

This is what we get for the sum in the limit as $h \to 0$. Hence, the total work done is $5,000$ foot–pounds.

Now let us consider what we have done, and identify the essential steps. Integration means "making whole". This refers to the "adding up" procedure towards the end of the problem, and we used an antiderivative – namely $x^2/2$, which is an antiderivative of x – to do the sum in the limit $h \to 0$. (This is the passage from the Riemann sum to the integral). However, before you can "make something whole", you have to first "take it apart", and all higher dimensional integration problems begin like this. This is where the most cleverness is usually required. Depending on how you choose to "slice" your problem at the beginning, you can be faced with integration problems of quite different degrees of difficulty. So although we say we are studying integration in this part of the course, most of our effort will be focused on disintegration – we want to do this in a thoughtful, careful way that facilitates the integration steps at the end. We begin with some simple problems in which the most obvious sort of disintegration works just fine.

1.2: Integrals on two variables

Consider a region Ω in $\mathbb{R}^2$. To be concrete, suppose that Ω is the closed unit disk in $\mathbb{R}^2$. That is, Ω consists of all points (x, y) satisfying

$$x^2 + y^2 \leq 1 \ . \tag{1.2}$$

Suppose that we have a sheet of metal lying in this region, and it has a mass density of $f(x, y)$ mass units per area units. (Grams per square centimeter if you like). What is the total weight of the sheet of metal?

If the mass density function $f(x, y)$ were constant, we could use the formula

$$\text{mass} = \text{massdensity} \ \times \ \text{area} \ . \tag{1.3}$$

If x and y are measured in centimeters, the area of Ω is π square centimeters, and so if the density were a uniform 1 gram per square centimeter, the total weight would be π grams.

But suppose that the disk of metal is thinner near the center, and has the mass density

$$f(x,y) = x^2 + y^2 \ .$$

What would be the total weight in this case? Less, clearly, but how much less?

The way forward is to disintegrate the the disk into small bits in which the mass density is effectively constant, and then to apply the formula (1.3) to each of these. This gives us the mass of each of the pieces. Since the mass of the whole is the sum of the mass of the parts, all we need do is to add up all of these masses, and make the disk whole again. This is the integration phase.

To disintegrate the disk, we chop it up on a rectangular grid. Let Δx be the spacing between the vertical grid lines and let Δy be the spacing between the horizontal grid lines. Most of the disk is covered by rectangular "tiles" of area $\Delta x \Delta y$. There are some tiles with more complicated shapes around the boundary, but these will account for a small percentage of the disk if both Δx and Δy are very small. Hence. lets ignore these for now, and focus on the rectangular tiles. In each of these, the mass density does no vary much – at least when both Δx and Δy are very small – so it makes sense to talk about the value of the mass density in the little tile. For each such tile, we have that the mass is

$$(\text{mass density in the tile}) \ \times \ \Delta x \Delta y \ .$$

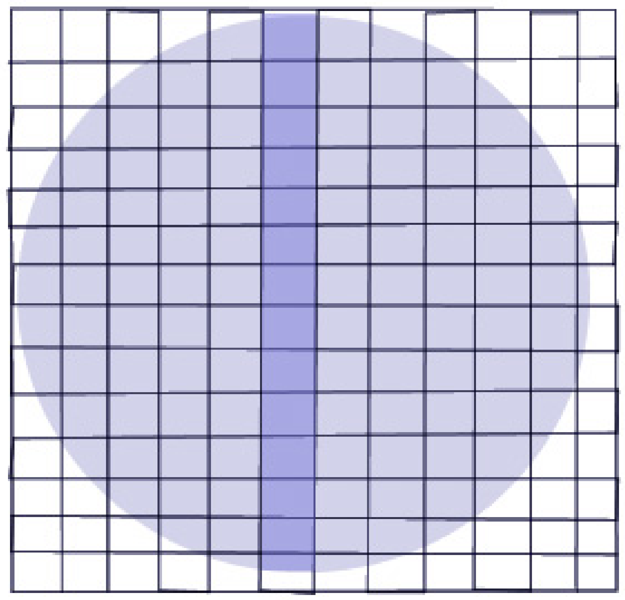
Hence the total mass is

$$\sum_{\text{little tiles}} (\text{mass density in the tile}) \ \times \ \Delta x \Delta y \ . \tag{1.4}$$

Now we are ready for the integration phase. We can add up the terms in the sum in any order we like – addition is commutative, and the sum is finite. There are two very natural ways to proceed:

• *We and add up the contributions from the tiles in each column, and then we can add up the sums for each column, or we can add up the contributions from the tiles in each row, and then we can add up the sums for each row.*

Lets first add up the contributions from the tiles in each column. Suppose that there are M columns, labeled by $j = 1, 2, \ldots, M$.



Then we have

$$\begin{aligned} &\sum_{\text{little tiles}} (\text{mass density in the tile}) \ \times \ \Delta x \Delta y \\ &= \sum_{i=1}^{N} \left(\sum_{\text{little tiles in column } j} (\text{mass density in the tile}) \ \times \ \Delta x \Delta y \right) \\ &= \sum_{i=1}^{N} \left(\sum_{\text{little tiles in column } j} (\text{mass density in the tile}) \ \times \ \Delta y \right) \Delta x \end{aligned} \tag{1.5}$$

If x_j is the x coordinate of, say, the middle of the jth column, then the inner sum, namely

$$\sum_{\text{little tiles in column } j} (\text{mass density in the tile}) \ \times \ \Delta y$$

is the Reimann sum for the integral

$$\int_{a(x_j)}^{b(x_j)} f(x_j, y)\mathrm{d}y \ ,$$

where $a(x_j)$ is the y coordinate at the bottom of the jth column and $b(x_j)$ is the y coordinate at the top of the jth column. In the case at hand, from the equation $x^2+y^2 = 1$ at the boundary of the region, we have

$$a(x_j) = -\sqrt{1-x_j^2} \qquad \text{and} \qquad b(x_j) = \sqrt{1-x_j^2}$$

and so, in more concrete terms, our integral is

$$\int_{-\sqrt{1-x_j^2}}^{\sqrt{1-x_j^2}} f(x_j, y)\mathrm{d}y \ .$$

For any fixed value of x_j, this is a garden variety definite integral in the single variable y. Doing it, we get

$$x_j^2 y + y^3/3\Big|_{-\sqrt{1-x_j^2}}^{\sqrt{1-x_j^2}} = 2x_j^2\sqrt{1-x_j^2} + (2/3)(1-x_j^2)^{3/2} \ .$$

Going back to (1.5), we see that, upon replacing the inner sum by the integral to which it corresponds (when viewed as a Riemann sum), we have that the total mass is

$$\sum_{i=1}^{N}\left(2x_j^2\sqrt{1-x_j^2} + (2/3)(1-x_j^2)^{3/2}\right)\Delta x \ .$$

Since the values of x in the disk range from -1 to 1, this is the Reimann sum for

$$\int_{-1}^{1}(2x^2(1-x^2)^{1/2} + (2/3)(1-x^2)^{3/2})\mathrm{d}x \ .$$

Using the trigonometric substitution $x = \sin(\theta)$, this is easily evaluated, and the answer is $\pi/2$.

In the limit as Δx and Δy both tend to zero, the approximations that we made in replacing sums by integrals, and choosing values in the small tiles, etc, all become increasingly negligible, and so this is the exact value for the total mass.

This problem makes for a good case study of the process of disintegration and integration. Here is the general result. Let Ω be some region given by inequalities of the form

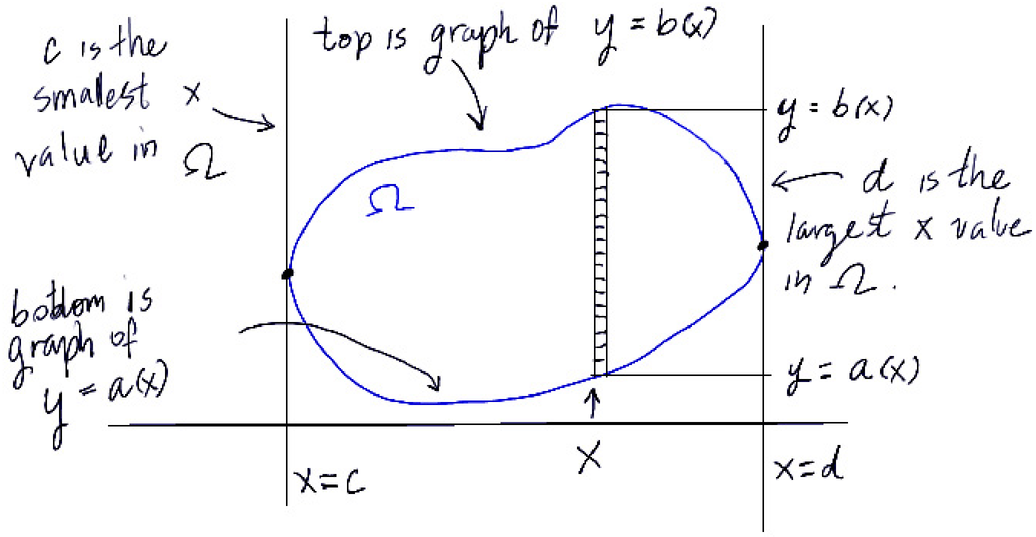
$$a(x) \le y \le b(y) \qquad\qquad c \le x \le d \ .$$

Let f be a continuous function defined on Ω.

We define the *area integral* $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ to be

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{\text{tile diameter}\to 0}\left(\sum_{\text{little tiles}} (\text{value of } f \text{ in the tile}) \ \times \ (\text{area of tile})\right) \ . \tag{1.6}$$

The following diagram show Ω, and the tiles in the column above x.



We do not require that the disintegration be done in such a way that all tiles have the same area, but when we talk about tile diameter going to zero, we mean the largest diameter of all the tiles in our "mesh".

Suppose, as n this diagram, every vertical line "slices" Ω in a single line segment (or misses it altogether). That is, the vertical line through x intersects Ω in an interval $[a(x), b(x)]$ of y values, or else is empty. If Ω were more complicated, the intersection could consist of several intervals, or worse. But for now, let's consider this nice case. Then, using a rectangular mesh, as above, and summing over columns first, we are led to the following formula for $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$:

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d \left(\int_{a(x)}^{b(x)} f(x,y)\mathrm{d}x \right) \mathrm{d}y \ . \tag{1.7}$$

In the inner integral, x is just a parameter, not a variable, so that it is a garden variety integral in the single variable y. Once it is done, y is eliminated, and what remains is a a garden variety integral in the single variable x. do that, and you are done.

Example 1 (Computation of an area integral) Let Ω be the region bounded above by the parabola $y = x^2 - 1$, and below by the parabola $y = x^2 - 1$. Let $f(x,y) = x^2 + 2xy$. Let's compute $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$.

First notice that every vertical line intersects Ω in a single segment, so we can use (1.7) and the disintegration into little rectangular blocks. We need to determine c and d, and $a(x)$ and $b(x)$.

Notice that the points (x,y) in Ω are those that satisfy

$$x^2 - 1 \le y \le 1 - x^2 \ .$$

The two parabolas meet at $x = \pm 1$, so $c = -1$ is the smallest x value in Ω, and $d = 1$ is the largest x value in Ω. The upper part of the boundary is $y = 1 - x^2$, so we take $a(x) = 1 - y^2$. The lower part of the boundary is $y = x^2 - 1$, so we take $b(x) = x^2 - 1$.

Hence, (1.7) becomes

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \int_{-1}^1 \left(\int_{x^2-1}^{1-x^2} (x^2 + 2xy)\mathrm{d}y \right) \mathrm{d}x \ . \tag{1.8}$$

In the inner integral, x is a parameter, and y is the variable of integration. So we treat x as a constant and have

$$\int_{x^2-1}^{1-x^2} (x^2 + 2xy)\mathrm{d}y = (x^2y + xy^2)\Big|_{x^2-1}^{1-x^2} = 2x^2(1-x^2) \ .$$

Now (1.8) reduces to

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \int_{-1}^{1} 2x^2(1-x^2)\mathrm{d}x = 8/15 \ .$$

Doing the sum in (1.6) by summing over rows first amounts to interchanging the roles of x and y so that we have the alternate formula

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\mathrm{d}y \right) \mathrm{d}x \ . \tag{1.9}$$

provided each horizontal line intersects Ω in a single line segment (or not at all). This time, c is the smallest y value in Ω, and d is the largest y value in Ω, and for values of y in between, the intersection of Ω with the horizontal line through y is the line segment $[a(y), b(y)]$. In the inner integral, y is just a parameter, not a variable, so that it is a garden variety integral in the single variable x. Once it is done, x is eliminated, and what remains is a a garden variety integral in the single variable y. do that, and you are done.

Example 2 (Alternate computation of an area integral) Let Ω be the region bounded above by the parabola $y = x^2 - 1$, and below by the parabola $y = x^2 - 1$. Let $f(x,y) = x^2 + 2xy$. Let's compute $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$, but this time by integrating first in x. We can do this using (1.9) since every horizontal line intersects Ω in a single segment, . We need to determine c and d, and $a(y)$ and $b(y)$.

For values of y with $0 \le y \le 1$, the interval is given by the equation for the upper parabola, and for values of y with $-1 \le y \le 0$, the interval is given by the equation for the lower parabola. Hence we break the region into two pieces, the upper region Ω_u and the lower region Ω_ℓ. It is clear from the definition that

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \int_{\Omega_u} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{\Omega_\ell} f(x,y)\mathrm{d}x\mathrm{d}y \ ,$$

so we just need to compute these separately.

In Ω_ell, the endpoints of the segment obtained by slicing the region horizontally at height y are given by the equation $y = x^2 - 1$. solving for x, we find $x = \pm\sqrt{1+y}$. Hence in Ω_ℓ we have

$$-\sqrt{1+y} \le x \le \sqrt{1+y}$$

so we take $a(y) = -sqrt1 + y$ and $b(y) = \sqrt{1+y}$, and clearly $c = -1$ and $d = 0$. Then (1.9) gives us

$$\int_{\Omega_u} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{-1}^{0} \left(\int_{-\sqrt{1+y}}^{\sqrt{1+y}} (x^2 + 2xy)\mathrm{d}x \right) \mathrm{d}y \ .$$

Doing the inner integral, treating y as constant,

$$\int_{-\sqrt{1+y}}^{\sqrt{1+y}} (x^2 + 2xy)\mathrm{d}x = (x^3/3 + x^2y)\Big|_{-\sqrt{1+y}}^{\sqrt{1+y}} = (2/3)(1+y)^{3/2} \ .$$

Hence

$$\int_{\Omega_u} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{-1}^{0} (2/3)(1+y)^{3/2}\mathrm{d}y = 4/15\ .$$

For the upper region, the endpoints of the segment obtained by slicing the region horizontally at height y are given by the equation $y = 1 - x^2$. solving for x, we find $x = \pm\sqrt{1-y}$. Hence in Ω_u we have

$$-\sqrt{1-y} \le x \le \sqrt{1-y}$$

so we take $a(y) = -\sqrt{1-y}$ and $b(y) = \sqrt{1-y}$, and clearly $c = 0$ and $d = 1$. Then (1.9) gives us

$$\int_{\Omega_u} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \left(\int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x^2 + 2xy)\mathrm{d}x \right) \mathrm{d}y\ .$$

Doing the inner integral, treating y as constant,

$$\int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x^2 + 2xy)\mathrm{d}x = (x^3/3 + x^2 y)\Big|_{-sqrt1-y}^{sqrt1-y} = (2/3)(1-y)^{3/2}\ .$$

Hence

$$\int_{\Omega_u} f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 (2/3)(1-y)^{3/2}\mathrm{d}y = 4/15\ .$$

Finally, we have

$$\int_{\Omega} f(x,y)\mathrm{d}x\mathrm{d}y = 4/15 + 4/15 = 8/15\ ,$$

which is what we found before.

We get the same value both ways – as we had to – but notice that the first way was a lot easier. How much calculation one has to do will depend very much on how one goes about the the disintegration and integration processes. Both involve choices – how do we slice? Do we add up columns first, or rows? So far we have only discussed slicing the region Ω into rectangles, but there are many other choices to consider. And as we have seen, the order in which we choose to integrate out the variables will affect the amount of work we must do.

Problems

Problem 1 Let $f(x,y) = x^3 y$, and let Ω be the region that lies to the right of the parabola $x = y^2$, and below the line $2y = -x$. Check that both horizontal lines and vertical lines intersect Ω in either a single interval, or the empty set.

(a) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ in terms of integrated integrals using (1.7). (You will need two iterated integrals as in Example 2).

(b) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ as an integrated integral using (1.9). (You will need only one iterated integral, as in Example 1).

(c) Evaluate one of the integrals.

Problem 2 Let $f(x,y) = x^2 y^2$, and let Ω be the region that lies inside both of the circles $(x-1)^2 + y^2 = 4$ and $(x+1)^2 + y^2 = 4$. Check that both horizontal lines and vertical lines intersect Ω in either a single interval, or the empty set.

(a) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ as an integrated integral using (1.7).

(b) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ as an integrated integral using (1.9).

(c) Evaluate one of the integrals.

Problem 3 Let $f(x,y) = x^2y^2$, and let Ω be the region that lies below the parabola $y = 4-(x-2)^2$ and above the x axis. Check that both horizontal lines and vertical lines intersect Ω in either a single interval, or the empty set.

(a) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ as an integrated integral using (1.7).

(b) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ as an integrated integral using (1.9).

(c) Evaluate one of the integrals.

Problem 4 Let $f(x,y) = xy$, and let Ω be the region bounded by the lines $y = x$, $y = 3x$, and $y = 5x-6$. Check that both horizontal lines and vertical lines intersect Ω in either a single interval, or the empty set.

(a) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ in terms of integrated integrals using (1.7).

(b) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ in terms of integrated integrals using (1.9).

(c) Evaluate one of the integrals.

Problem 5 Let $f(x,y) = x^2+y^2$, and let Ω be the region bounded by the lines $y = -x$, $y = x$ and $y = 5-2x$. Check that both horizontal lines and vertical lines intersect Ω in either a single interval, or the empty set.

(a) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ in terms of integrated integrals using (1.7).

(b) Write down $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ in terms of integrated integrals using (1.9).

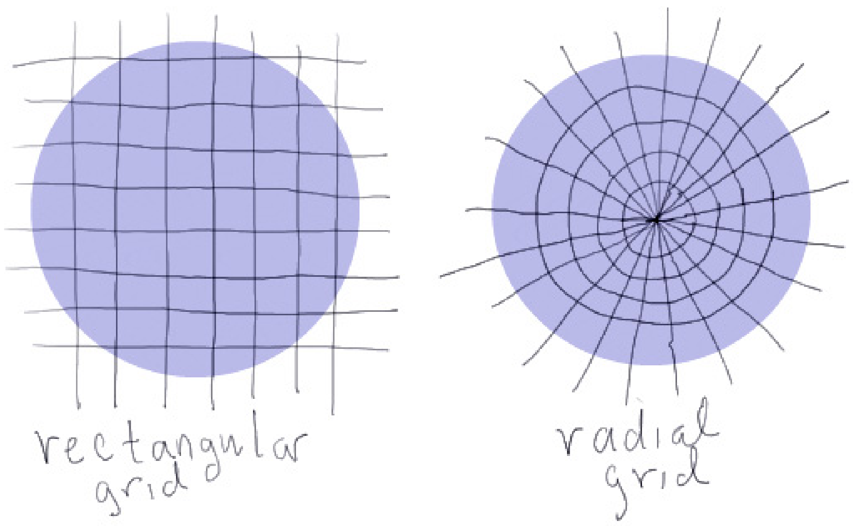
(c) Evaluate one of the integrals.

Section 2: Changing the variables of integration in $\mathbb{R}^2$

2.1: Other slicing strategies – How would you cut a cake?

How would you cut a cake? That would probably depend on the shape of the cake. If the cake were rectangular, cutting it into square or rectangular slices would seem sensible. But it it were round, you would probably cut it into wedges. Making cuts along the radii, it is easy to divide a round cake into, say, a dozen equal pieces. This is not so easy if you only make cuts parallel to the lines in a rectangular grid.

When we are disintegrating a region Ω in $\mathbb{R}^2$, it can be quite advantageous, for some of the same reasons, to slice using a grid of radii and concentric circles:

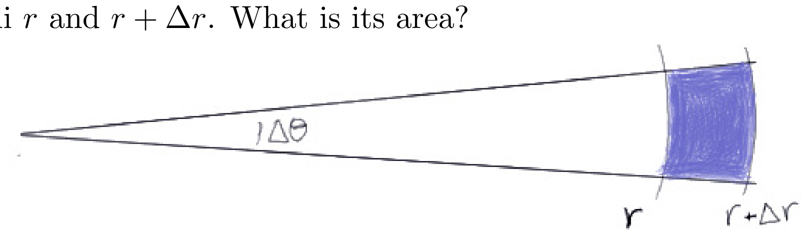


The basic formula that defines the integral is

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{\text{tile diameter}\to 0}\left(\sum_{\text{little tiles}} (\text{value of } f \text{ in the tile}) \ \times\ (\text{area of tile})\right) . \tag{1.1}$$

We can use this with tiles of any shape we find convenient. Of course, a *sine qua non* of convenience is that we have a simple formula for the area of the tiles. This is one of the things that is so attractive about rectangular tiles: The area of a rectangular tile of width Δx and height Δy is simply $\Delta x \Delta y$.

Now consider a "keystone" shaped tile that comes from a wedge of angle $\Delta\theta$, and lies between the radii r and $r + \Delta r$. What is its area?



The keystone shaped tile can be thought of as the part of the circular wedge with opening angle $\Delta\theta$ and radius $r + \Delta r$, that lies outside the circular wedge of the same angle and radius r. Subtracting the smaller wedges area from the larger, we are left with the area of the tile.

A circular wedge of opening angle θ and radius R is the fraction $\theta/(2\pi)$ of a disk of radius R. (That is how we measure angles – by the fraction of the circumference they subtend). The area of the disk is πR^2, and so the area of the wedge is

$$\frac{\theta}{2\pi}\pi R^2 = \frac{\theta R^2}{2} \ .$$

The area of our keystone is therefore the difference of the area of two wedges:

$$\frac{\Delta\theta(r+\Delta r)^2}{2} - \frac{\Delta\theta r^2}{2} = r\Delta r\Delta\theta + \frac{\Delta\theta(\Delta r)^2}{2} \ .$$

When both Δr and $\Delta\theta$ are very small, the second term on the right is negligible compared to the first, and so

$$\text{area of keystone tile} \ \approx \ r\Delta r\Delta\theta \ .$$

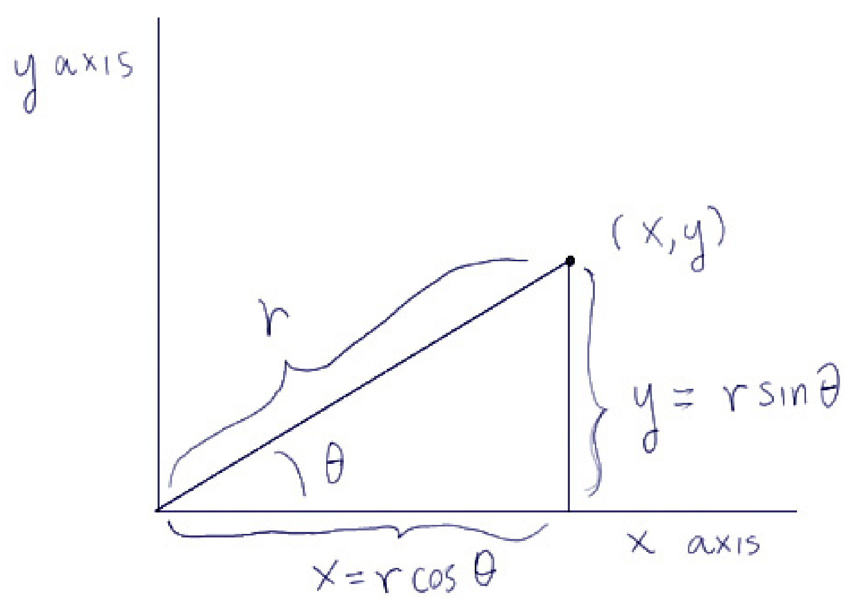
As Δr and $\Delta\theta$ diminish, the error in this approximation diminishes in the sense that it becomes a negligibly small fraction of the main term, $r\Delta r\Delta\theta$. This is getting smaller too, but not so fast.

2.2: Polar coordinates

The grid that we are using to cut the plane into keystone shaped tiles is based on the polar coordinate system, and we will need to be able to convert between polar coordinates – r and θ – and Cartesian coordinates – x and y – to use this slicing strategy. As you see, the keystone tiles are naturally indexed by r and θ. Therefore, it is natural to express the integrand $f(x,y)$ in these terms.

This is easy: If we measure θ counterclockwise from the positive x axis, and if r is the distance from the origin, then

$$x = r\cos(\theta) \qquad \text{and} \qquad y = r\sin(\theta) \ . \tag{1.2}$$



In particular,

$$x^2 + y^2 = r^2 \ . \tag{1.3}$$

By definition, r is always positive. It has a geometric meaning – distance form the origin – and distances cannot be negative.*

Think of (1.2) as a *dictionary* for translating Cartesian coordinates into polar coordinates. You might think we would be more interested in formulas for r and θ in terms of x and y. We do have a formula for r in terms of x and y, namely (1.3), and we could solve (1.2) for θ, but actually, what we really need is just (1.2) itself:

• *To translate a function f from Cartesian into polar terms, define a new function $g(r,\theta)$ by*

$$g(r,\theta) = f(r\cos(\theta), r\sin(\theta)) \ .$$

The meaning of this is that if $\mathbf{x}$ is any point in $I\!R^2$, then we can evaluate f at $\mathbf{x}$ by substituting the polar coordinates of $\mathbf{x}$ into g.

Example 1 (Translating a function into polar terms) Let $f(x,y) = x^2y$ Then

$$g(r\theta) = (r\cos(\theta))^2 r\sin(\theta) = r^3\cos^2(\theta)\sin(\theta) \ .$$

If only everything were so easy!

2.3: Polar coordinates and integration in $I\!R^2$

Let's apply our considerations to the problem of evaluating $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$.

If we cut Ω into keystone shaped tiles using polar coordinates, and then want to compute

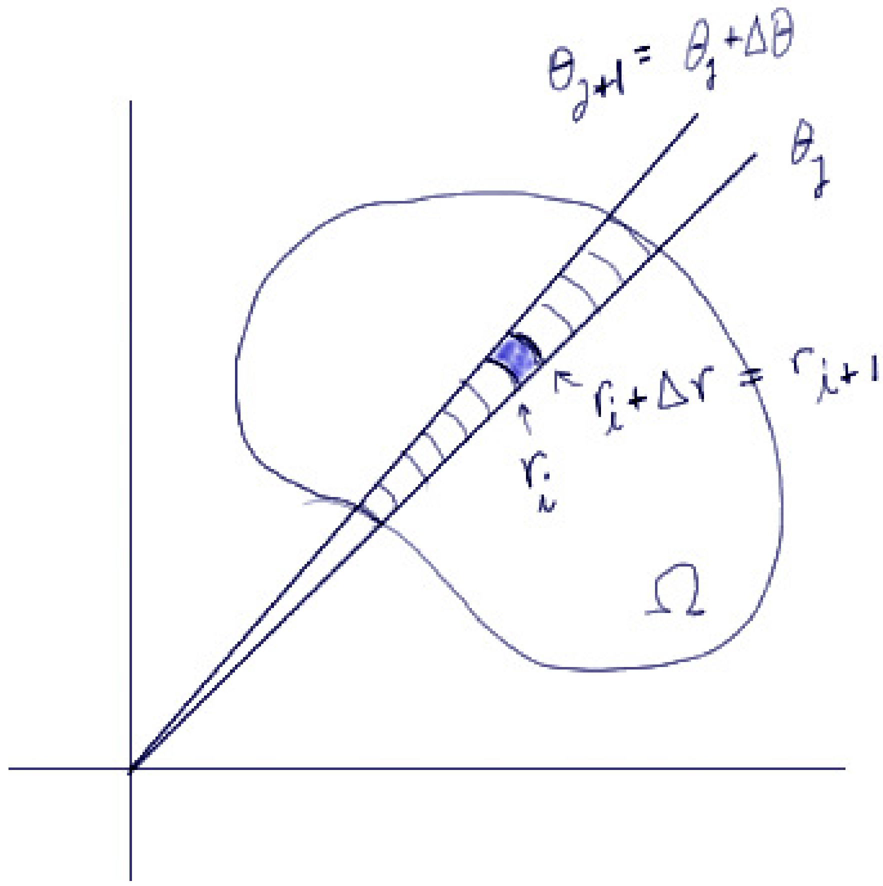
$$\sum_{\text{little tiles}} (\text{value of } f \text{ in the tile}) \ \times \ (\text{area of tile}) \ ,$$

we must chose an order in which to add up the contributions from each tile. The one that is most often convenient is to add up all of the contributions from each wedge, and then add up the subtotals for each wedge.

The following diagram show a region Ω, with the wedge cut through Ω by the radii at θ_j and θ_{j+1}, where some small $\Delta\theta$ has been fixed, and $\theta_j = j\Delta\theta$. For example, suppose we choose some large integer N, and let $\Delta\theta = 2\pi/N$, so that we divide $I\!R^2$ into N wedges with opening angle $\Delta\theta$.

This wedge has been further broken up into keystone tiles by cutting along circular arc of radius r_i where some small value of Δr has been chosen and $r_i = \Delta r$.

* You may have worked with polar coordinates before using a different convention in which a negative value of r meant that the point would lie at distance $|r|$ from the origin in the opposite direction, namely, the one corresponding to $\theta + pi$. There are some advantages to this in drawing curves, so this convention is occasionally used, However, there are disadvantages as well that are more important here. In the example that follow, we will use the positivity of r several times, and you will see this.



We now organize our summation as follows: For each j, we hold j fixed, and sum up the contributions form each of the tiles in the jth wedge. That is, we sum over i first, holding j fixed. Then we add up these subtotals into the grand total by summing on j:

$$\sum_{\text{little tiles}} (\text{value of } f \text{ in the tile}) \times (\text{area of tile}) =$$

$$\sum_{j=1}^{N} \left(\sum_{\text{little tiles in wedge } j} g(r_i, \theta_j) r_i \Delta r \Delta\theta \right) =$$

$$\sum_{j=1}^{N} \left(\sum_{\text{little tiles in wedge } j} g(r_i, \theta_j) r_i \Delta r \right) \Delta\theta \ ,$$

where $r_i = i\Delta r$ is the ith value of r used in our grid.

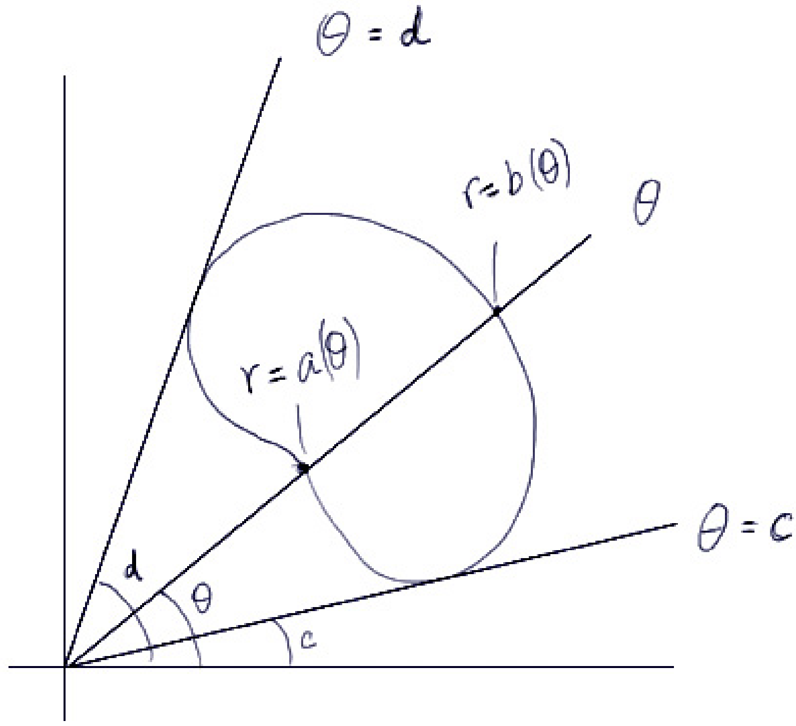
Notice that the inner sum,

$$\sum_{\text{little tiles in wedge } j} g(r_i, \theta_j) r_i \Delta r$$

is just the Riemann sum for an integral. If $a(\theta_j)$ is the smallest value of r in Ω that lies in the jth wedge, and if $b(\theta_j)$ is the largest value of r in Ω that lies in the jth wedge, then this is a Riemann sum for

$$\int_{a(\theta_j)}^{b(\theta_j)} g(r,\theta_j) r\mathrm{d}r \ . \tag{1.4}$$

Here is a diagram showing $a(\theta)$ and $b(\theta)$.



The diagram also shows the smallest and largest values of θ for which the ray in direction θ intersects the region Ω. These are denoted c and d. Clearly $a(\theta)$ and $b(\theta)$ are only defined for $c \le \theta \le d$.

The value of the integral (1.4) depends on θ_j of course. For $c \le \theta_j \le d$, call it $G(\theta_j)$. There are no keystones to worry about for other values of θ_j, so our sum reduce to

$$\sum_{\text{little tiles}} (\text{value of } f \text{ in the tile}) \ \times \ (\text{area of tile}) =$$

$$\sum_{j \text{ such that } c \le \theta_j \le d} G(\theta_j)\Delta\theta \ ,$$

and this is a Riemann sum for $\int_c^d G(\theta)\mathrm{d}\theta$. Altogether, we have the formula

$$\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y = \int_c^d \left(\int_{a(\theta)}^{b(\theta)} g(r,\theta) r\mathrm{d}r \right) \mathrm{d}\theta \ .$$

Example 2 (An integral in polar coordinates Let $f(x,y) = x^2$, and let Ω be the region bounded by the circle

$$(x-1)^2 + y^2 = 1 \ . \tag{1.5}$$

Compute $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$.

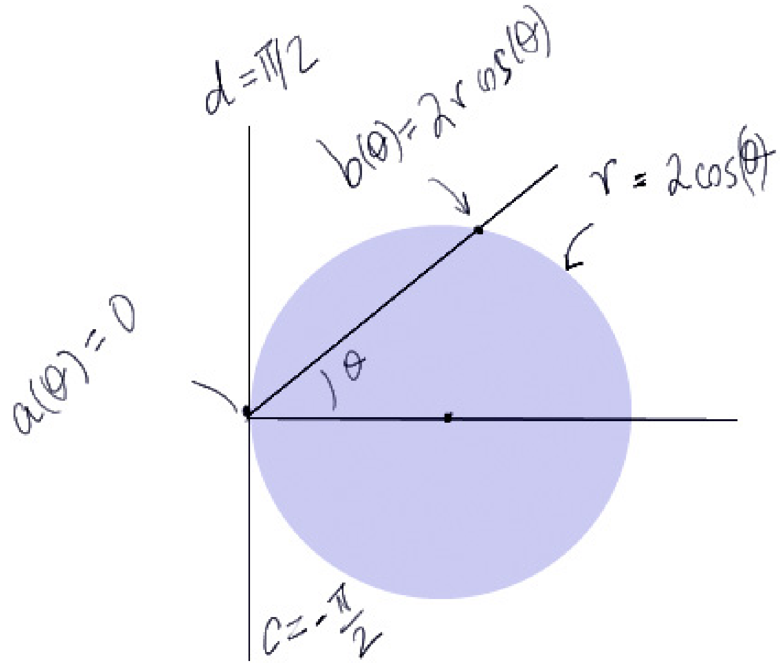
The region is a circle, and though it is not centered, we might expect it to have a nice description in polar coordinates. Let's see. Simplifying, the equation reduces to

$$x^2 + y^2 = 2x\ .$$

Using (1.2), this translates into $r^2 = 2r\cos(\theta)$. Since r is strictly positive except at the origin, (1.6) reduces to

$$r = 2\cos(\theta) \tag{1.7}$$

. This equation is very simple, and will enable us to find simple expressions for $a(\theta)$ and $b(\theta)$, and also c and d. To do this, draw a diagram, and label the boundary of Ω with the equation that specifies it:



Notice that the formula (1.7) would give a negative value for r in the second and the quadrants, but has a positive value in the first and fourth quadrants. this tells us that the region Ω "lives" in these quadrants, and $c = -\pi/2$ and $d = \pi/2$. You see this also in the picture, but drawing a picture is not always so easy. Hence it is important to see how the values of c and d can be read off of (1.7).

As for $a(\theta)$ and $b(\theta)$, draw in a ray at angle θ, as in the diagram. It enters Ω at $r = 0$, and leave through the boundary with the equation $r = 2\cos(\theta)$. Hence $a(\theta) = 0$, and $b(\theta) = 2\cos(\theta)$. That takes care of the limits. The rest is easy.

Translating the integrand using (1.2),

$$g(r\theta) = (r\cos(\theta))^2 = r^2\cos^2(\theta)\ .$$

Therefore,

$$\begin{aligned}
\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y &= \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos(\theta)} r^2\cos^2(\theta) r\delta r\right)\delta\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos(\theta)} r^3\delta r\right)\cos^2(\theta)\delta\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(\frac{2^4\cos^4(\theta)}{4}\right)\cos^2(\theta)\delta\theta \\
&= 4\int_{-\pi/2}^{\pi/2}\cos^6(\theta)\delta\theta\ .
\end{aligned}$$

The problem is now reduced to a single variable integral with explicit limits, and for our purposes, is done. We will regard all such integrals as "trivial" for the purposes of this course. It is a non trivial matter to make this reduction, and Maple cannot help you. Once you are here, you can type the trivial integral into Maple, and get the final numerical answer, which is $5\pi/4$.

Example 3 (Area enclosed by the Bernoulli lemiscate The Bernoulli lemiscate is the curve given by

$$(x^2 + y^2)^2 = 2(x^2 - y^2) \ . \tag{1.8}$$

This is the "infinity symbol". Let's compute the enclosed area, which is

$$\int_\Omega 1 \mathrm{d}x \mathrm{d}y \ .$$

That is, to get an area, the integrand should just be 1. (Reflect on the definition to make sure this is clear).

Since the integrand features $x^2 + y^2$ which will reduce to r^2 in polar coordinates, we will translate (1.8) into polar terms, hoping for something nice. As it stands, (1.8) is pretty awful. It is a quartic equation, and solving to find either x as a function of y, or y as a function of x, is a daunting proposition, and a big mess. So, let's try something else. At this point, the only option is polar coordinates, so try that.

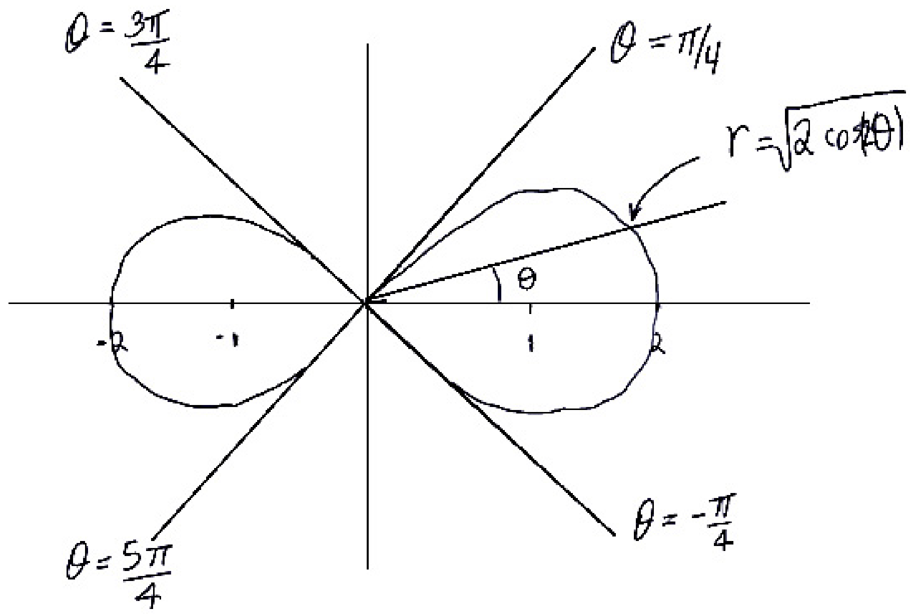
Using (1.2), (1.8) becomes $r^4 = 2r^2(cos^2(\theta) - \sin^2(\theta))$. Then using the double angle formulas, and dividing through by r^2, (1.8) reduces to

$$r^2 = 2\cos(2\theta) \ . \tag{1.9}$$

This is sweet: The variables are separated with a clear functional dependence. You can also see from (1.9) that the right hand side is negative unless

$$-\pi/4 \le \theta \le \pi/4 \qquad \text{or} \qquad 3\pi/4 \le \theta \le 5\pi/4 \ .$$

Hence the curve described by (1.9), or equivalently (1.8), "lives" in these two angular sectors. Here is a rough sketch:



You could produce such a sketch by evaluating $r = \sqrt{2\cos(2\theta)}$ for a few values of θ in the range $-\pi/4 \le \theta \le \pi/4$, drawing those points in, and connecting the dots. You do not have to know in advance that our equation describes the infinity symbol.

Notice that the equation (1.8) only involves x^2 and y^2, so if (x, y) satisfies the equation, so do the mirror image points

$$(-x, y) \qquad (x, -y) \qquad (-x, -y) \ .$$

That is, we can see from the equation that the region is *symmetric* under reflection about the x–axis and about the y–axis. This is not so evident form the rough sketch, but that is O.K.; the equations make it clear.

Because of the symmetry, the area in the first quadrant is exactly one fourth of the total. Hence we can take $c = 0$ and $d = \pi/4$, and remember to multiply by 4 when we have finished integrating.

From the diagram, you see that $a(\theta) = 0$ and $b(\theta) = \sqrt{2\cos(2\theta)}$, so the integral we need to do is

$$\int_1^{\pi/4} \left(\int_0^{\sqrt{2\cos(2\theta)}} 1r\mathrm{d}r \right) \mathrm{d}\theta \ .$$

The inner integral is trivial, and we are left with

$$\int_1^{\pi/4} \cos(2\theta)\mathrm{d}\theta = 1/2 \ .$$

Multiplying by 4, the area of the infinity symbol, scaled to cut the x–axis at $r = 2$, is 4.

Problems

Problem 1 Let $f(x, y) = y$, and let Ω be the region inside both of the circles

$$(x-1)^2 + (y-1)^2 = 2 \qquad \text{and} \qquad (x+1)^2 + (y-1)^2 = 2 \ .$$

Compute $\int_\Omega fx, y)\mathrm{d}x\mathrm{d}y$.

Problem 2 Consider the region enclosed by the curve

$$x^2 + y^2 = (x^2 + y^2 - x)^2 \ .$$

Show that in polar coordinates, this curve is given by

$$r = 1 + \cos(\theta) \ .$$

Sketch the curve, and compute the area it encloses.

Problem 3 Consider the closed curve given in polar terms by $r = \sin^3(\theta)$. Sketch this curve, and compute the area enclosed.

Problem 4 Consider the closed curve given in polar terms by $r = 1 + \sin(2\theta)$. Sketch this curve, and compute the area enclosed.

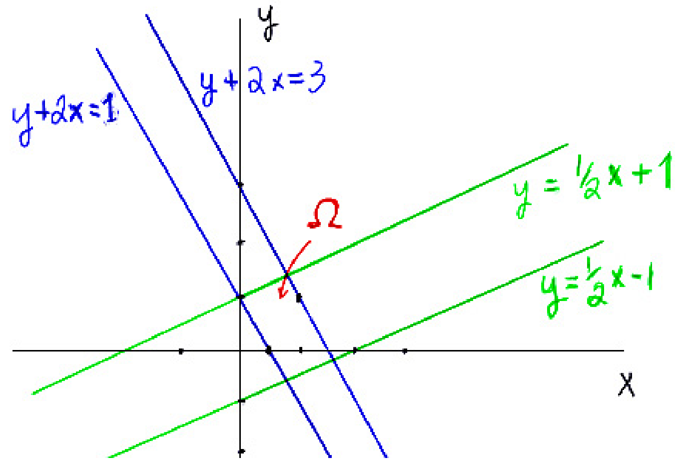
Section 3: Jacobians and changing variables of integration in $\mathbb{R}^2$

3.1: Letting the boundary of Ω determine the disintegration strategy

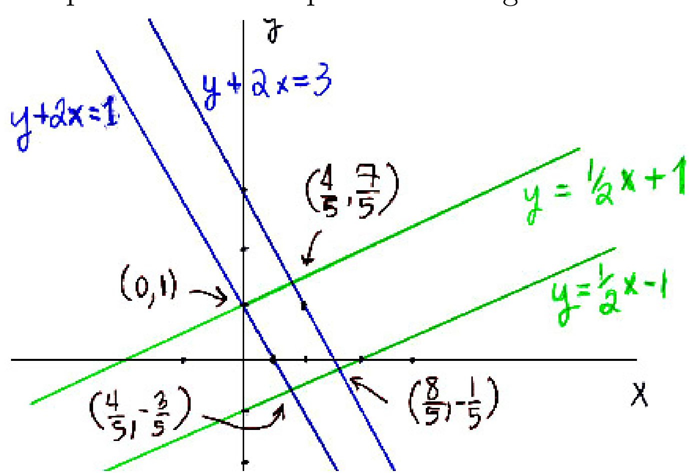
Consider the problem of computing $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$ where $f(x,y) = y$, and where Ω is bounded by the 4 lines

$$y + 2x = 1 \qquad y + 2x = 3 \qquad 2y - x = -2 \qquad 2y - x = 2\ .$$

Here is a picture of the region:



To find the limits of integration, we next work out the coordinates of the vertices by solving the systems of equations for each pair of crossing lines:



If we integrated in x first we would need to break Ω in the three separate subregions for

$$-\frac{3}{5} \le y \le -\frac{1}{5} \qquad -\frac{1}{5} \le y \le 1 \qquad 1 \le y \le \frac{7}{5}$$

since in each of these regions we need a different formula for $a(y)$ or $b(y)$ – horizontal segments at height y begin and end on the same bounding line in only when y stays in one of these ranges.

If we integrated in y first, we could do better: We would only need to break Ω in the two separate subregions for

$$0 \le x \le \frac{4}{5} \qquad \frac{4}{5} \le x \le \frac{8}{5}$$

since in each of these regions we need a different formula for $a(x)$ or $b(x)$ – vertical segments at x begin and end on the same bounding line only when x stays in one of these ranges.

So, if these were our only choices, certainly we would integrate in y first. However, there is something better we can do. Instead of disintegrating Ω using a grid composed of lines parallel to the axes, let's disintegrate Ω using a grid of lines paralell to the bounding lines.

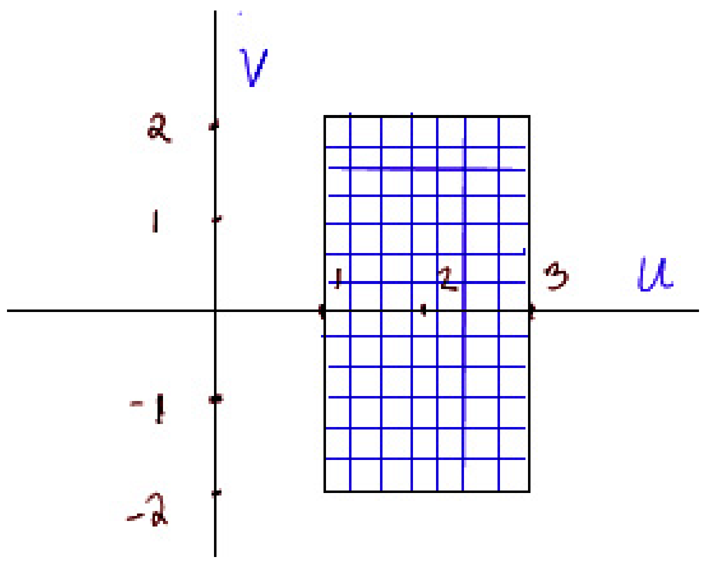
To do this, define new variables

$$u = y + 2x \qquad v = 2y - x \; . \tag{1.1}$$

Then in terms of these variables, the region Ω is bounded by

$$u = 1 \quad u = 3 \qquad v = -2 \qquad v = 2 \; .$$

In the u, v plane, this is a rectangle with sides parallel to the axes, and we can easily divide it up along a rectangular grid.



The jth vertical line in this grid is the line

$$u = 1 + j\Delta u \tag{1.2}$$

where Δu is the horizontal spacing in the grid, and the ith horizontal line in the grid is

$$v = -2 + i\Delta v \tag{1.3}$$

where Δv is the vertical spacing in the grid. (We are ordering the lines left to right and bottom to top respectively).

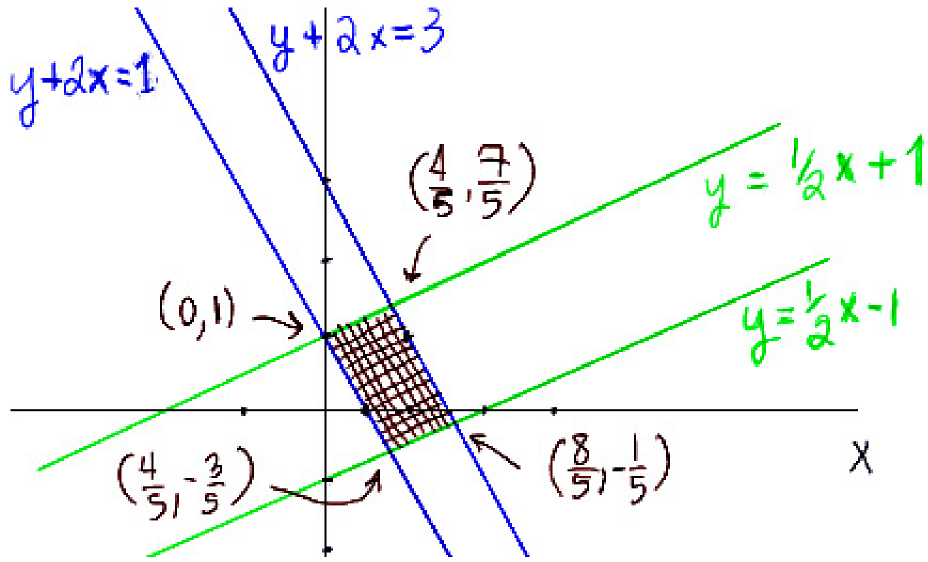
Using (1.1) to express (1.2) and (1.3) in terms of x and y we get

$$y + 2x = 1 + j\Delta u \tag{1.4}$$

and

$$2y - x = -2 + i\Delta v \tag{1.5}$$

This gives us two sets of parallel, evenly spaced lines in the x, y plane that divide Ω up into similar parallelogram shaped tiles.



We will now use this grid to do our disintegration of Ω. Using these tiles, we will compute

$$\int_{\Omega} f(x,y)\mathrm{d}x\mathrm{d}y = \lim_{\text{tile diameter}\to 0}\left(\sum_{\text{little tiles}} (\text{value of } f \text{ in the tile}) \ \times \ (\text{area of tile})\right) . \tag{1.6}$$

Now, each of these tiles in Ω corresponds to a tile in the u, v plane, and so we can enumerate the tiles in our disintegration of Ω using an enumeration of the corresponding tiles in our disintegration of the rectangle $1 \le u \le 3$, $-2 \le v \le 2$ is the u, v plane. To do this we need to answer two questions:

• Given a tile with $u_j \le u \le u_j + \Delta u$ and $v_i \le v \le v_i + \Delta v$, what is the value of $f(x, y)$ at some point (x, y) is the corresponding tile in the x, y plane?

• Given a tile with $u_j \le u \le u_j + \Delta u$ and $v_i \le v \le v_i + \Delta v$, what is the area of the corresponding tile in the x, y plane?

To answer the first question, we solve (1.1) for x and y as functions of u and v. We can write (1.1) as

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} .$$

Inverting, we find,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} . \tag{1.7}$$

In other words,

$$x = \frac{2u - v}{5} \qquad \text{and} \qquad y = \frac{2v + u}{5} .$$

Now that we have formulas for $x(u, v)$ and $y(u, v)$; that is, for x and y as functions of u and v, we can define

$$g(u, v) = f(x(u, v), y(u, v)) .$$

Clearly, evaluating g at (u, v) gives the same values as evaluating f at (x, y) if (x, y) and (u, v) are related by the transformation in (1.7). This answers the first question:

• *Given a tile with $u_j \le u \le u_j + \Delta u$ and $v_i \le v \le v_i + \Delta v$, $g(u_j, v_i)$ is the value of $f(x, y)$ at a corner of the corresponding tile in the x, y plane.*

It is now easy to answer the second question. The tiles in the x, y plane are the images of tiles in the u, v plane under the linear transformation in (1.7). The "maginification factor" of a 2×2 matrix J is $|\det(J)|$, which is to say that the image of a region under J will have an area that is $|\det(J)|$ times as large as the area of the original region.

Applying this to one of our tiles in the u, v plane, notice that the initial area is just $\Delta u \Delta v$. Hence, whith J denoting the matrix in (1.7), the area of the corresponding tile in the x, y plane is

$$|\det(J)|\Delta u \Delta v .$$

In the case at hand, $|\det(J)| = 1/5$, and so the area of a tile in our grid in the x, y plane is

$$\frac{1}{5}\Delta u \Delta v .$$

Going back to (1.6), we now have

$$\begin{aligned} \int_\Omega f(x, y)\mathrm{d}x\mathrm{d}y &= \lim_{\Delta u, \Delta v \to 0} \left(\sum_{i,j} (g(u_j, v_i) \ \times \ \left(\frac{1}{5}\Delta u \Delta v\right)\right) \\ &= \lim_{\Delta u, \Delta v \to 0} \left(\sum_i \left(\sum_j \frac{1}{5}(g(u_j, v_i)\Delta u\right) \ \Delta v\right) \ . \end{aligned} \tag{1.8}$$

You recognize the Riemann sums for

$$\int_{-2}^{2} \left(\int_1^3 \frac{1}{5} g(u, v)\mathrm{d}u \right) \mathrm{d}v .$$

In the case at hand, $g(u,v) = (2v+u)/5$, and so

$$\begin{aligned}\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y &= (1/25)\int_{-2}^{2}\left(\int_1^3 (2v+u)\mathrm{d}u\right)\mathrm{d}v\\ &= (1/25)\int_{-2}^{2}\left((2vu+u^2/2)\Big|_{u=1}^{u=3}\right)\mathrm{d}v\\ &= (1/25)\int_{-2}^{2}(4v+4)\,\mathrm{d}v\\ &= 16/25\ .\end{aligned}$$

What is the lesson to be drawn from this example? It is that:

• *By using a disintegration scheme that "respected" the equations defining the boundaries of Ω, we were able to avoid breaking up Ω into subregions that would have to be handled separately, and we got very simple limits of integration – constants in this case.*

The moral is to treat the boundary conditions with respect.

3.2: What if Ω is not bounded by straight lines?

The strategy developed in the last section can be applied even when the boundary of Ω is not given by straight lines. There is very little adaptation required if we remember the main idea of the Calculus: *Up close enough, all nice functions are linear for all practical purposes.*

Let's consider a second example.

Consider the region Ω in the upper right quadrant bounded by

$$xy = 1 \qquad xy = 3 \qquad 2x = y \qquad x = 2y\ .$$

Let's compute its area.

Two of the bounding curves are arcs of hyperbolas, and the other two are lines. However, notice that if we introduce

$$u = xy \qquad \text{and} \qquad v = y/x\ , \tag{1.9}$$

we can write the equations for the boundary as

$$u = 1 \qquad u = 3 \qquad v = 2 \qquad v = 1/2\ .$$

Again, this is simply a rectangle in the u,v plane.

Think of (1.9) as defining a transformation from the x,y plane to the u,v plane. What we would like to have instead is the *inverse transformation* from the u,v plane to the x,y plane, since we can then use this transformation to "transplant" a grid on the rectangle

$$1 \le u \le 3 \qquad \text{and} \qquad 1/2 \le v \le 2 \tag{1.10}$$

onto Ω, just as we did in the last problem. All we have to do is to solve (1.9) for x and y as functions of u and v. From (1.9), we see that $uv = y^2$. Since Ω is in the upper right quadrant, $y > 0$, and so $y = \sqrt{uv}$. Next, $x^2 = u/v$, and since $x > 0$, $x = \sqrt{u/v}$. This gives us

$$x = \sqrt{u/v} \qquad \text{and} \qquad y = \sqrt{uv} \tag{1.11}$$

To emphasize that we are going to think about this as a transformation from $I\!R^2$ to $I\!R^2$,* we introduce $\mathbf{F}$ by

$$\mathbf{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} \sqrt{u/v} \\ \sqrt{uv} \end{bmatrix} .$$

Then we can write (1.11) as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) .$$

Now consider a small tile with $u_j \le u \le u_j + \Delta u$ and $v_i \le v \le v_i + \Delta v$ in the u, v plane. The image of this tile is a slightly distorted parallelogram with vertices at

$$\mathbf{F}(u_j, v_i) \qquad \mathbf{F}(u_j + \Delta u, v_i) \qquad \mathbf{F}(u_j, v_i + \Delta v) \qquad \mathbf{F}(u_j + \Delta u, v_i + \Delta v) .$$

The distortion will be slight to the extent that Δu and Δv are small – everything nice looks linear up close enough.

To compute the area of this parallelogram, we first apply the approximation

$$\mathbf{F}(\mathbf{u}) = \mathbf{F}(\mathbf{u}_0) + J_{\mathbf{F}}(\mathbf{u}_0)(\mathbf{u} - \mathbf{u}_0)$$

with the basepoint $\mathbf{u}_0 = \begin{bmatrix} u_j \\ v_i \end{bmatrix}$, which is the lower left vertex of the tile in the u, v plane. We have:

$$\begin{aligned}
\mathbf{F}(u_j, v_i) &= \mathbf{F}(\mathbf{u}_0) \\
\mathbf{F}(u_j + \Delta u, v_i) &\approx \mathbf{F}(\mathbf{u}_0) + J_{\mathbf{F}}(\mathbf{u}_0)\begin{bmatrix} \Delta u \\ 0 \end{bmatrix} \\
\mathbf{F}(u_j, v_i + \Delta v) &\approx \mathbf{F}(\mathbf{u}_0) + J_{\mathbf{F}}(\mathbf{u}_0)\begin{bmatrix} 0 \\ \Delta v \end{bmatrix} \\
\mathbf{F}(u_j + \Delta u, v_i \Delta v) &\approx \mathbf{F}(\mathbf{u}_0) + J_{\mathbf{F}}(\mathbf{u}_0)\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix}
\end{aligned}$$

In this approximation, the parallelogram is the image of the rectangle with vertices

$$(0,0) \qquad (\Delta u, 0) \qquad (0, \Delta v) \qquad (\Delta u, \Delta v)$$

---

* Actually, from the upper right quadrant of $I\!R^2$ to the upper right quadrant of $I\!R^2$

under the linear transformation induced by $J_{\mathbf{F}}(\mathbf{u}_0)$, and then translated by $\mathbf{F}(\mathbf{u}_0)$.

Translation has no affect on area, and the linear transformation multiplies the area of the original rectangle, namely $\Delta u \Delta v$ by the factor $|\det(J_{\mathbf{F}}(\mathbf{u}_0))|$. Therefore, using the notation introduced above:

- *The image under* $\mathbf{F}$ *of the tile with* $u_j \le u \le u_j + \Delta u$ *and* $v_i \le v \le v_i + \Delta v$ *is a tile in the* x, y *plane whose area is*

$$|\det(J_{\mathbf{F}}(\mathbf{u}_0))|\Delta u \Delta v$$

up to an error that is vanishingly small percentagewise as Δu *and* Δv *both go to zero.*

Everything is pretty much as it was in our last example, except that now $|\det(J_{\mathbf{F}}(\mathbf{u}))|$ is not a constant. Computing, we find

$$J_{\mathbf{F}}(\mathbf{u}) = \frac{1}{2}\begin{bmatrix} u^{-1/2}v^{-1/2} & u^{-1/2}v^{-3/2} \\ u^{-1/2}v^{1/2} & u^{1/2}v^{-1/2} \end{bmatrix}$$

Therefore,

$$|\det(J_{\mathbf{F}}(\mathbf{u}))| = \frac{1}{2uv} \ .$$

This gives us a formula for the area of the image of a small tile at u, v, namely

$$\frac{1}{2uv}\Delta u \Delta v \ .$$

This is often referred to at the formula for the *area element.*

In an area computation, our integrand is 1, which requires no translation. However, we can go ahead and say what we would do if the integrand were some function $f(x, y)$. We would define $g(u, v)$ by $g(\mathbf{u}) = f(\mathbf{F}(\mathbf{u}))$. The definition is such that if (x, y) corresponds to (u, v) under the transformation $\mathbf{F}$, then $f(x, y) = g(u, v)$.

Going back to the basic formula (1.6), we have

$$\text{area of } \Omega = \int_\Omega 1\mathrm{d}x\mathrm{d}y = \lim_{\text{tile diameter}\to 0}\left(\sum_{\text{little tiles}} 1 \ \times \ (\text{area of tile})\right) \ . \tag{1.12}$$

Using the tiles induced by the transformation $\mathbf{F}$ through the regular rectangular grid on the rectangle (1.10), we get the Riemann sums for

$$\int_{1/2}^{2}\left(\int_1^3 \frac{1}{2uv}\mathrm{d}u\right)\mathrm{d}v \ .$$

The two integrals are now easily worked out with the result that the area is ln(6).

3.3: Substitution in two variables

What we have worked out in the previous section is a *substitution*, or *change of variables* formula for integrals in two variables.

The general picture is this: Suppose that $\mathbf{F}$ is an invertible transformation from some subset of $I\!R^2$ to another subset of $I\!R^2$. Think of it as transforming the u, v plane to the x, y plane so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) .$$

Let f be a continuous function on some region Ω that is contained in the domain of definition of $\mathbf{F}$. Since $\mathbf{F}$ is invertible, we can define a region Ξ in the u, v plane by

$$(u, v) \text{ belongs to } \Xi \qquad \Longleftrightarrow \qquad \mathbf{F}(u, v) \text{ belongs to } \Omega .$$

Since the transformation $\mathbf{F}$ is invertible, it sets up a one–to–one correspondence between points in Ξ and points Ω so that any disintegration of Ξ induces a disintegration of Ω.

Consider the image of a rectangular tile of width Δu and height Δv sitting in Ξ with its lower left corner at (u, v). As we have explained above, the area of the corresponding tile in Ω is well approximated by

$$|\det(J_{\mathbf{F}}(u, v)|\Delta u \Delta v .$$

Therefore, if we define $g(\mathbf{u}) = f(\mathbf{F}(\mathbf{u}))$, we will have

$$\int_\Omega f(x, y)\mathrm{d}x\mathrm{d}y = \int_\Xi g(u, v)\ |\det(J_{\mathbf{F}}(u, v)|\mathrm{d}u\mathrm{d}v . \tag{1.13}$$

Another common notation for expressing this is to write $\mathrm{d}^2\mathbf{x}$ in place of $\mathrm{d}x\mathrm{d}y$, and $\mathrm{d}^2\mathbf{u}$ in place of $\mathrm{d}u\mathrm{d}v$. We can just use the definition of $g(\mathbf{u})$ together with this notation to write

$$\int_\Omega f(\mathbf{x})\mathrm{d}^2\mathbf{x} = \int_\Xi f(\mathbf{F}(\mathbf{u}))\ |\det(J_{\mathbf{F}}(\mathbf{u})|\mathrm{d}^2\mathbf{u} . \tag{1.14}$$

This may be compared to the formula for substitution, or change of variables, in one dimension. Suppose $F(u)$ is a differentiable function on R, and we define $x = F(u)$, Then if f is any continuous function of one variable, we have

$$\int_a^b f(x)\mathrm{d}x = \int_c^d f(F(u))F'(u)\mathrm{d}u \tag{1.15}$$

$a = F(c)$ and $b = F(d)$.

Notice that the determinant of the Jacobian of F is the higher dimensional replacement for for the derivative F' in the one dimensional formula. However, in the one dimensional formula, there is no absolute value sign. Why is this?

Suppose that $c < d$ as usual, and suppose that F is invertible. Even though F is invertible, it might be decreasing, so that $a = F(c) > F(d) = b$. In this case F' is negative, but we can cancel this minus sign with the minus sign that comes from swapping the limits on the left. In other words, if we define

$$\tilde{a} = \min\{F(c)\ ,\ F(d)\} \qquad \text{and} \qquad \tilde{b} = \max\{F(c)\ ,\ F(d)\}$$

so that $\tilde{a} < \tilde{b}$ and $[\tilde{a}, \tilde{b}]$ defined an interval, we could rewrite (1.15) as

$$\int_{[\tilde{a},\tilde{b}]} f(x)\mathrm{d}x = \int_{[c,d]} f(F(u))|F'(u)|\mathrm{d}u \tag{1.16}$$

and now we get a formula that looks even more like (1.14).

In writing the simpler formula (1.15), we are taking advantage of the fact that the real numbers are ordered. There is no natural ordering of the points in a region of $I\!R^2$, and so there is no natural analog of "switching the limits of integration".

It is important to stress that the formula (1.15) is valid even if F is not a one–to–one function, but no so (1.16), and not so its higher dimensional analog (1.14). For example, if as u sweeps through $[c, d]$, the interval $F(u)$ sweeps though the interval $[\tilde{a}, \tilde{b}]$ three times, then you would need a factor of 3 on the left in (1.16) for it to be valid. Similar rules counting the number of times the image of Ξ covers Ω under $\mathbf{F}$ would allow us to consider transformations that are not invertible. Here, we will only work with invertible transformations; this suffices for the solution of many practical problems.

Now that we have the change of variables formula (1.14), we can put it to work directly, without explicitly going through considerations of "little tiles". That is not to say that the "little tiles" way of thinking is expendable in any way. Among other things, it is essential for setting up integrals that arise in word problems – the only way they arise in real life.

However, let us close this subsection with some examples of (1.14) in action. We will focus on how one finds $\mathbf{F}$ and hence Ξ.

Actually, in practice one is led first to a formula for $\mathbf{F}^{-1}$, since this gives u and v as functions of x and y:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{F}^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) . \tag{1.17}$$

Usually, staring at the definition of Ω, we come up with some definitions of u and v in terms of x and y; that is, with an explicit formula for the transformation $\mathbf{F}^{-1}$ in (1.16). The first order of business then is to solve this system of equations to find x and y as functions of u and v, or, in other words, to find $\mathbf{F}$.

Example 1 (Using the change of variables formula in $I\!R^2$) Let Ω be the region in the upper right quadrant between the curves

$$x = \frac{1}{y^2} \qquad \text{and} \qquad x = \frac{4}{y^2}$$

and between the curves

$$y = x^2 \qquad \text{and} \qquad y = 4x^2 \ .$$

Lets compute $\int_\Omega (x^2 + y^2)\mathrm{d}^2\mathbf{x}$.

If we define

$$u = xy^2 \qquad \text{and} \qquad v = y/x^2 \ , \tag{1.18}$$

the region is described by

$$1 \le u \le 4 \qquad \text{and} \qquad 1 \le v \le 4 \ . \tag{1.19}$$

To find $\mathbf{F}$, we just need to solve (1.18) for x and y in terms of u and v. We can eliminate x by forming $u^2v = y^3$, so $y = u^{2/3}v^{1/3}$. Next, we can eliminate y by forming $uv^{-2} = x^3$, so that $x = u^{1/3}v^{-2/3}$. This gives us

$$\mathbf{F}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u^{1/3}v^{-2/3} \\ u^{2/3}v^{1/3} \end{bmatrix} \ .$$

With this definition of $\mathbf{F}$, Ξ is the rectangle (1.19).

Next, we compute

$$J_{\mathbf{F}} = \frac{1}{3}\begin{bmatrix} u^{-2/3}v^{-2/3} & -2u^{1/3}v^{-5/3} \\ 2u^{-1/3}v^{1/3} & u^{2/3}v^{-2/3} \end{bmatrix} \ .$$

Therefore,

$$\det(J_{\mathbf{F}}(\mathbf{u})) = \frac{5}{9}v^{-4/3} \ .$$

Next, with $f(x,y) = x^2 + y^2$,

$$f(\mathbf{F}(u,v)) = u^{2/3}v^{-4/3} + u^{4/3}v^{2/3} \ .$$

Hence, from (1.14), we have

$$\int_\Omega f(\mathbf{x})\mathrm{d}^2\mathbf{x} = \int_\Xi (u^{2/3}v^{-4/3} + u^{4/3}v^{2/3})\frac{5}{9}v^{-4/3}\mathrm{d}^2\mathbf{u}$$

and since Ξ is jst the rectangle (1.19), this becomes

$$\frac{5}{9}\int_1^4 \left(\int_1^4 (u^{2/3}v^{-8/3} + u^{4/3}v^{-2/3})\mathrm{d}u\right)\mathrm{d}v \ .$$

Problems

Problem 1 Let $f(x,y) = y$, and let Ω be the region bounded by $x + y = 2$, $x + y = 4$, $x^2y = 1$ and $x^2y = 2$. Compute $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$.

Problem 2 Let Ω be the region bounded by $x^4 + y^4 = 1$. Compute its area. (Use symmetry to conclude that the area is 4 times the area of the piece in the upper right quadrant, and set up an integral to compute that). Leave your answer in the form of an explicit integral over one variable. If you do this in the way that is intended, you will be left with what is known as an *elliptic integral*. They come up all the time, and Maple has the means to deal with then programmed in.

Problem 3 Let $f(x,y) = xy$, and let Ω be the region bounded by $xy = 1$, $xy = 2$, $y/x = 1$ and $y/x = 2$. Compute $\int_\Omega f(x,y)\mathrm{d}x\mathrm{d}y$.