Final Exam for Calculus III for CS-Majors, Math 2605A1-2
Mai 1, 2003

Name:

Student ID number:

This test is to be taken without calculators and notes of any sorts. The allowed time is 2 **hours and** 50 **minutes.** Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

I: (7+5+5 points) Consider the function

$$f(x, y) = x^2 + y^2 - 4x^2y^2 \ .$$

a) Find the gradient and the Hessian of this function.

b) Find all the critical points of f and sketch a few of the level curves nearby them.

c) Sketch a few level curves of this function.

II (15 points) Find all the points (x, y) with $x^2 + 3y^2 \leq 1$ where

$$e^{x^2 - y^2}$$

has its maximum and minimum values.

III: (15 points) Find an approximate solution for the system of nonlinear equations

$$x + y^3 = 3$$

$$x^2 + 2y^2 = 4$$

using Newton's method. Start from the point $(1, 1)$ and run the iteration twice. Then plug this approximate solution back into the equation to see how precise it is.

IV: (5+5+5+5 points) Consider the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} .$$

a) Diagonalize A.

b) Find the Schur decomposition of A.

c) Find the Householder reflection that maps the first column of A to a multiple of $\vec{e}_1$.

d) Find the QR factorization of A.

V: (6+5+4 points) a) Compute the singular value decomposition of the matrix

$$\frac{1}{5\sqrt{2}}\begin{bmatrix} 2 & -10 \\ 11 & -5 \end{bmatrix}.$$

b) Compute $A_{(1)}$, the best rank one approximation for A.

c) What is the norm of $\|A - A_{(1)}\|$?

VI: True or False (4 points each): a) Every matrix A, with distinct eigenvalues can be written in the form $A = QDQ^T$ where Q is a rotation and D is diagonal.

b) It takes $n - 1$ Householder reflections to find the QR decomposition of an $n \times n$ matrix A.

c) Every matrix can be written in the form $A = VDU^T$ where V and U are rotations.

d) Every matrix with distinct eigenvalues can be diagonalized.

e) There is an iterative method for diagonalizing symmetric matrices and it takes finitely many iterations to arrive at its diagonal form.

VII: (8+7 points) a) Find all the equilibrium points of the system below and discuss their stability:

$$\dot{x} = -x + xy \ , \dot{y} = 2y + xy \ .$$

b) Solve the initial value problem

$$\dot{x} = 6x + 4y \ , \dot{y} = 4y \ , x(0) = 1, y(0) = 0 \ .$$

VIII: (8+5 points) a) Find the rotation matrix Q whose angle of rotation, θ, is given by $\cos(\theta) = 3/5$ and whose axis of rotation is given by

$$\frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \ .$$

b) Find the axis and angle of rotation of Q^3. (Think before going into a long computation!)

IX: (5 points) Calculate the area of the region Ω that is bounded below by the curve $y = x^4$ and above by the curve $y = x^2$.

X: (10 points) Calculate

$$\int_\Omega x \, \mathrm{d}x \mathrm{d}y$$

where Ω is the region that is bounded by the curves

$$xy = 1 \ , xy = 3 \ , \frac{x}{y} = 2 \text{ and } \frac{x}{y} = 5 \ .$$