Solution to Preptest 1C

Problem 1

The gradient is

$$\nabla f(x,y) = \frac{1}{(1+x^2+y^2)^3}(2x(1-x^2+3y^2), -2y(1+3x^2-y^2)) \ .$$

$$\nabla f(-1,1) = \frac{3}{4}(-1,-1)$$

a) Tangent plane

$$z = -\frac{3}{4}(x+1) + (y-1)$$

b)

$$\nabla f(1,0) = (0,0)$$

and there is no tangent line.

c)

$$(x,y) = (5/2, \pm 1/\sqrt{2})$$

Problem 2 a) Critcal points are

$$(0,0) \ , \ (0,\pm 1) \ , \ (\pm 1, 0) \ .$$

b) c) At $(0,0)$: The Hessian is

$$\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

and the type is a saddle.

At $(\pm 1, 0)$: The Hessian is

$$\begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$$

and the type is a maximum

At $(0, \pm 1)$: The Hessian is

$$\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

and the type is a Minimum

Plot:

Problem 3 a)

$$[\,-1/3, -1\,]$$

b) $|F(\mathbf{x_0})| = \sqrt{8}$. $|F(\mathbf{x_1})| = \sqrt{(26/27)^2 + (2/3)^2} \approx 32/27$ which is an improvement.

Extra credit

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \quad .$$