Pratice Test 2A for Math2605, Fall 2004

Problem 1

Find the maximum value of the function

$$\frac{1}{1+x^2+(y-1)^2}$$

on the set given by all pairs (x, y), such that $x^2 - y^2 \geq 1$. Find all the points where the maximal value is attained.

Critical points: The gradient of the function is

$$-\frac{2}{1+x^2+(y-1)^2}\begin{bmatrix} x \\ y-1 \end{bmatrix}$$

and $(0, 1)$ is the only critical point. However, this point is not in the region $x^2 - y^2 \geq 1$. Thus there are no critical points in the region $x^2 - y^2 \geq 1$.

Thus we have to find the maximum value of the function subject to the constraint $x^2 - y^2 = 1$. Use the method of Lagrange multiplier we get

$$-\frac{2}{1+x^2+(y-1)^2}\begin{bmatrix} x \\ y-1 \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ -y \end{bmatrix}$$

which has to be solved together with the equation $x^2 - y^2 = 1$. Thus we have to solve the system of equations

$$x^2 - y^2 = 1$$
$$-x = \lambda x(1 + x^2 + (y-1)^2)$$
$$y - 1 = \lambda y(1 + x^2 + (y-1)^2)$$

If $\lambda = 0$ then $x = 0$ and $y = 1$ which, as we said before, does not satisfy the first equation. Hence $\lambda \neq 0$ and

$$-xy(1 + x^2 + (y-1)^2) = x(y-1)(1 + x^2 + (y-1)^2)$$

$x \neq 0$ (Why?) and hence $-y = y - 1$ or $y = 1/2$. Hence

$$x^2 = \frac{5}{4}$$

and there are two points where the maximum is attained

$$(\frac{\sqrt{5}}{2}, \frac{1}{2}) \ , \ (-\frac{\sqrt{5}}{2}, \frac{1}{2})$$

The value there is

$$\frac{2}{5} \ .$$

Problem 2

a) Apply one step of the Jacobi iteration for diagonalizing the matrix

$$\begin{bmatrix} 2 & 0.1 & 1 \\ 0.1 & 4 & 0.2 \\ 1 & 0.2 & 2 \end{bmatrix}$$

Make sure that you do it in such a fashion that the new matrix is almost diagonal.

The largest off diagonal element is 1 and hence we have to diagonalize

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

which has the eigenvalues/eigenvectors

$$3 \ , \ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ ; \ 1 \ , \ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The Givens rotation is

$$G = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now,

$$B := G^T A G = \begin{bmatrix} 3 & \frac{0.3}{\sqrt{2}} & 0 \\ \frac{0.3}{\sqrt{2}} & 4 & \frac{0.1}{\sqrt{2}} \\ 0 & \frac{0.1}{\sqrt{2}} & 1 \end{bmatrix}$$

b) Give upper and lower bounds on the eigenvalues as accurately as you possibly can.

We apply the 'Small Gershgorin disk' Theorem. Note that the smallest difference between the diagonals is $\delta(B) = 1$ and the maximum among the sum of the off diagonals is $r(B) = \frac{0.4}{\sqrt{2}}$. Clearly $\delta(B) < r(B)$ and hence

$$\frac{2r(B)^2}{\delta(B)} = 0.16 \ .$$

Hence we have the following bounds for the eigenvalues:

$$4.16 \geq \mu_1 \geq 3.84 \ , \ 3.16 \geq \mu_2 \geq 2.84 \ , \ 1.16 \geq \mu_3 \geq 0.84 \ .$$

Problem 3

Consider the matrix

$$A = \begin{bmatrix} 2 & -6 \\ 2 & 9 \\ -8 & -6 \end{bmatrix} .$$

a) Find the singular value decomposition for this matrix.

$$A^T A = \begin{bmatrix} 72 & 54 \\ 54 & 153 \end{bmatrix} = 9 \begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix}$$

The eigenvalues and eigenvectors are

$$9 \times 20 \ , \ \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ ; \ 9 \times 5 \ , \ \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Hence we have

$$D = \begin{bmatrix} 6\sqrt{5} & 0 \\ 0 & 3\sqrt{5} \end{bmatrix}$$

and

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Finally

$$V = AUD^{-1} = \frac{1}{3} \begin{bmatrix} -1 & -2 \\ 2 & 1 \\ -2 & 2 \end{bmatrix}$$

b) Find the generalized inverse A^+ for this matrix

Calculate

$$A^+ = UD^{-1}V^T = \frac{1}{90} \begin{bmatrix} 7 & -2 & -10 \\ -6 & 6 & 0 \end{bmatrix}$$

c) Find the least square least length solution of the equation $A\mathbf{x} = \mathbf{b}$ where

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} .$$

$$A^+\mathbf{b} = -\frac{1}{18} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

d) Find $A_{(1)}$ the best rank one approximation of A.

$$A_{(1)} = 6\sqrt{5}\frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \frac{1}{\sqrt{5}} [\, 1 \ 2 \,] = \begin{bmatrix} -2 & -4 \\ 4 & 8 \\ -4 & -8 \end{bmatrix} .$$

Problem 4

a) Diagonalize the matrix

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} .$$

Eigenvalues $5, -1$ with the eigenvectors

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} , \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

$$O^T A O = \begin{bmatrix} 5 & -2 \\ 0 & -1 \end{bmatrix}$$

where

$$O = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

Another possibility would be

$$U^T A U = \begin{bmatrix} -1 & 2 \\ 0 & 5 \end{bmatrix}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

b) Find the Schur factorization of this matrix.

Problem 5: True or False : a) Every matrix A, with distinct eigenvalues can be written in the form $A = QDQ^T$ where Q is a rotation and D is diagonal. FALSE

b) It takes $n-1$ steps steps to reach a Schur factorization for an $n \times n$ matrix. TRUE

c) Every matrix can be written in the form $A = VDU^T$ where V and U are rotations. FALSE

d) Every matrix with distinct eigenvalues can be diagonalized. TRUE

e) There is an iterative method for diagonalizing symmetric matrices and it takes finitely many iterations to arrive at its diagonal form. FALSE