Solution to Practice Test 2B

Problem I: The critical points are found by solving

$$\begin{bmatrix} 3x^2 - 3y^2 + 8x \\ -6xy - 8y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Write the second equation as $(6x+8)y = 0$ This yields two alternatives: either $y = 0$ or $x = -4/3$. If $y = 0$ the first equation reads $(3x+8)x = 0$ and we get the solutions

$$(0,0) \ , \ (-\frac{8}{3}, 0) \ .$$

Next, if $x = -4/3$ the first equation reads $3y^2 + 16/3 = 0$ which does not have a real solution. Thus the above two points are the only critical points. The first is inside the unit disk and the second is not. Hence $(0,0)$ is the only critical point we have to consider.

Now we consider the boundary. Using Lagrange multipliers we have to solve the euqations

$$\begin{bmatrix} 3x^2 - 3y^2 + 8x \\ -6xy - 8y \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

and $x^2 + y^2 = 1$. Note that λ cannot be zero since that would amount to calculating critical points which we just did. Cross-multiplying leads to the equations

$$(3x^2 - 3y^2 + 8x)y = -(6x+8)yx \ , x^2 + y^2 - 1 = 0 \ .$$

Clearly $(1,0), (-1,0)$ are solutions. If $y \neq 0$ we can divide and get the new equations

$$(3x^2 - 3y^2 + 8x) = -(6x+8)x \ , x^2 + y^2 - 1 = 0 \ .$$

Eliminating y and solving for x we get two solutions $x_1 = -3/2$ which is not on the unit circle and $x = 1/6$ which leads to the points

$$(\frac{1}{6}, \pm\frac{\sqrt{35}}{6}) \ .$$

Now

$$f(0,0)=0\ ,\ f((\frac{1}{6},\frac{\sqrt{35}}{6})=-\frac{115}{27}=f((\frac{1}{6},-\frac{\sqrt{35}}{6})\ .$$

Thus f has a maximum at $(0,0)$ and a minimum at the other two points.

Problem II: a) $Off(A)=236$.

The submatrix is

$$\begin{bmatrix} 6 & 8 \\ 8 & -6 \end{bmatrix}$$

which has the eigenvalues $10,-10$ with the corresponding eigenvectors

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\ ,\ \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\ .$$

The Givens matrix G is

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & 0 & \frac{-1}{\sqrt{5}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & 0 & \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$G^TAG=\begin{bmatrix} 10 & \frac{7}{\sqrt{5}} & \frac{4}{\sqrt{5}} & 0 \\ \frac{7}{\sqrt{5}} & 3 & 0 & \frac{14}{\sqrt{5}} \\ \frac{4}{\sqrt{5}} & 0 & 1 & \frac{3}{\sqrt{5}} \\ 0 & \frac{14}{\sqrt{5}} & \frac{3}{\sqrt{5}} & -10 \end{bmatrix}$$

$$Off(G^TAG)=Off(A)-2\times 8^2=108\ .$$

Problem III: See the solution of the previous practice test.

Problem IV: Write the matrix as $A+tB$ where

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\ ,B=\begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 5 \\ 1 & 5 & 0 \end{bmatrix}$$

The eigenvalues of A with the corresponding eigenvectors are

$$2\ ,\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\ ;\ 0\ ,\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\ ;\ 2\ ,\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Using that $\mu(t) = \mu(0) + tv \cdot Bv + o(t)$ we get

$$\mu_1(t) = 2 + 2t + o(t) \ , \ \mu_2(t) = -2t + o(t) \ , \ \mu_3(t) = 2 + o(t) \ .$$

Calculate $G^T(A+tB)G$ where G is the matrix that has the eigenvectors of A as columns. Note that G is a rotation. This new matrix is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \frac{t}{\sqrt{2}} \begin{bmatrix} 4 & 0 & 6 \\ 0 & -4 & 4 \\ 6 & 4 & 0 \end{bmatrix}$$

the Gershgorin disks are the following: one centered at $2+0.4/\sqrt{2}$ with radius $0.6/\sqrt{2}$, one centered at $-0.4/\sqrt{2}$ with radius $0.4/\sqrt{2}$ and another one centered at 2 with radius $0.6/\sqrt{2}$. Each contains at least an eigenvalue. For every eigenvalue there is a Gershgorin disk which contains that eigenvlaue, and hence one eigenvalue satisfies $|\mu_2 + 0.4/\sqrt{2}| < 0.4/\sqrt{2}$ while the other two must sit between $2-0.6/\sqrt{2}$ and $2+1/\sqrt{2}$. More cannot be said.

Problem V:

$$F(x,y) = \begin{bmatrix} x^2 + xy + y^2 - 4 \\ x^3 - 2xy^2 + 2 \end{bmatrix}$$

and

$$F(1,1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The Jacobian is

$$\begin{bmatrix} 2x + y & x + 2y \\ 3x^2 - 2y^2 & -4xy \end{bmatrix}$$

At $(1,1)$ this is

$$\begin{bmatrix} 3 & 3 \\ 1 & -4 \end{bmatrix}$$

Its inverse is

$$\frac{1}{15} \begin{bmatrix} 4 & 3 \\ 1 & -3 \end{bmatrix}$$

Now

$$x_1 = x_0 - J_F^{-1}(x_0)F(x_0) = \frac{1}{15} \begin{bmatrix} 16 \\ 19 \end{bmatrix}$$

$$F(x_1) = \begin{bmatrix} \frac{21}{225} \\ \frac{4081}{3375} \end{bmatrix}$$

and

$$|F(x_1)| < 1.2128$$

Since

$$|F(1,1)| = \sqrt{2}$$

we have an improvement.