Pratice Test 3 for Math2605, Fall 2007

Problem 1

For the matrix

$$A = \begin{bmatrix} 4 & 8 \\ 3 & 2 \end{bmatrix}$$

a) Find the QR factorization of A.

The Householder reflection that maps the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ to the vector $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ is given by the matrix $Q = I - 2\mathbf{u}\mathbf{u}^T$ where $\mathbf{u} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. The result is

$$Q = \frac{1}{5} \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$$

and

$$R = QA = \frac{1}{5} \begin{bmatrix} 25 & 38 \\ 0 & 16 \end{bmatrix} \ .$$

b) The Schur factorization.

We have to find one eigenvalue and eigenvector, 8 and the vector $\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Next form the matrix

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

and compute

$$U^T A U =: T = \begin{bmatrix} 8 & 5 \\ 0 & -2 \end{bmatrix}$$

c) Compute e^{At}.

We start with the important observation that

$$e^{At} = U^T e^{Tt} U \ .$$

Then we note that

$$e^{Tt} = \begin{bmatrix} e^{8t} & a(t) \\ 0 & e^{-2t} \end{bmatrix}$$

and since $Te^{Tt} = e^{Tt}T$ we conclude that

$$a(t) = \frac{1}{2} \left(e^{8t} - e^{-2t} \right) \ .$$

Problem 2 Consider the rotation matrix

$$Q = \frac{1}{45} \begin{bmatrix} 40 & -5 & 20 \\ 13 & 40 & -16 \\ -16 & 20 & 37 \end{bmatrix}$$

a) Find the axis of rotation $\mathbf{u}$ and the angle of rotation θ.

Recall the two formulas, for the angle of rotation

$$\theta = \cos^{-1}\left(\frac{\mathrm{Tr}Q - 1}{2}\right)$$

and for the axis of rotation

$$\Omega_{\mathbf{u}} = 2\sin\theta\left(Q - Q^T\right) \ ,$$

where

$$\Omega_{\mathbf{u}} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}$$

Thus, in our example $\theta = \cos^{-1}(\frac{4}{5})$ and

$$\mathbf{u} = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

b) Find B so that $Q = e^{\theta B}$.

The matrix $B = \Omega_{\mathbf{u}}$ and hence equals

$$\frac{1}{3}\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

c) Find a rotation R so that $Q = R^2$.

Recall Euler's formula

$$Q(\theta) = \cos\theta I + (1 - \cos\theta)\mathbf{u}\mathbf{u}^T + \sin\theta\Omega_{\mathbf{u}}$$

we have to calculate $\cos(\theta/2)$ and $\sin(\theta/2)$ which follows from the formulas

$$\cos(\theta/2) = \sqrt{\frac{\cos\theta + 1}{2}} \ , \ \sin(\theta/2) = \sqrt{\frac{1 - \cos\theta}{2}}$$

which in our case yields

$$\cos(\theta/2) = \frac{3}{\sqrt{10}} \ , \sin(\theta/2) = \frac{1}{\sqrt{10}} \ .$$

Hence

$$R = \frac{3}{\sqrt{10}}I + (1 - \frac{3}{\sqrt{10}})\mathbf{u}\mathbf{u}^T + \frac{1}{\sqrt{10}}\Omega_{\mathbf{u}}$$

d) Find the family of rotations that interpolate the identity and Q.

This is just given by Euler's formula where the angle θ varies between 0 and $\cos^{-1}(4/5)$.

Problem 3 Consider the differential equation $x'' = x' - x + x^3$.
a) Write this equation as a first order system.

Set $x' = y$ and then $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and the differential equation reads

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}) = \begin{bmatrix} y \\ y - x + x^3 \end{bmatrix}$$

b) Find all the critical points.

Critical points are the same as equilibrium points, that is where bfF vanishes. This means that

$$y = 0 \ , \ -x + x^3 = 0 \ .$$

This yields the critical points

$$(0,0) \ , \ (1,0) \ , \ (-1,0) \ .$$

c) Find the type of the critical points and decide whether they are linearly stable or unstable.

We have to linearize $\mathbf{F}$ at the critical points, which means we have to calculate the Jacobi matrices:

$$J_{\mathbf{F}}((0,0)) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \ , \ J_{\mathbf{F}}((1,0)) = J_{\mathbf{F}}((-1,0)) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

d) Decide which ones are are stable or unstable for the nonlinear system and determine their possible types.

For the point $(0,0)$ the eigenvalues are complex with positive real part. The motion in the vicinity of this point is an unstable spiral. The other two are an unstable saddle since the two eigenvalues have the opposite sign.

Problem 4 Consider the curve

$$\mathbf{x}(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3 \ , \ y(t) = \frac{1}{2}t^2 + \frac{1}{3}t^3 \ , 1 \le t \le 2 \ .$$

a) Find the velocity for all values of t.

$$\mathbf{x}'(t) = \begin{bmatrix} t - t^2 \\ t + t^2 \end{bmatrix}$$

b) Find the unit tangent vector $\mathbf{T}(t)$.

$$\mathbf{T}(t) = \frac{1}{\sqrt{2(t^2 + t^4)}} \begin{bmatrix} t - t^2 \\ t + t^2 \end{bmatrix}$$

c) Find the length of the curve.

$$|\mathbf{x}'(t)| = \sqrt{2(t^2+t^4)}$$

and

$$L = \int_1^2 \sqrt{2(t^2+t^4)}dt = \sqrt{2}\int_1^2 \sqrt{(1+t^2)}tdt = \frac{1}{\sqrt{2}}\int_1^2 \sqrt{1+\tau}d\tau = \frac{1}{\sqrt{2}}\frac{2}{3}(1+\tau)^{3/2}|_1^2 =$$

$$\sqrt{6} - \frac{4}{3} \ .$$

d) Rewrite the curve in terms of the length parametrization s.

$$s(t) = \frac{\sqrt{2}}{3}(1+t)^{3/2} - \frac{4}{3}$$

Hence

$$t = \left(\frac{3}{\sqrt{2}}s + 2^{3/2}\right)^{2/3} - 1$$

Insert this into the formula for $\mathbf{x}(t)$ yields the curve in the length parametrization.
e) Find the curvature $\kappa(s)$.

Recall that

$$\mathbf{T}'(t) = v(t)\kappa(t)\mathbf{N}(t)$$

from which we glean the formula

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{v(t)} \ .$$

A straightforward calculation leads to

$$\mathbf{T}'(t) = \frac{1}{(1+t^2)^{3/2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} - t\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)$$

From which we see that

$$|\mathbf{T}'(t)| = \frac{1}{(1+t^2)}$$

and since $v(t) := |\mathbf{x}'(t)| = \sqrt{2(1+t^2)}$ we get that

$$\kappa(t) = \frac{1}{\sqrt{2}(1+t^2)^{3/2}} \ .$$