Solutions to practice test 3B

Problem I: a)

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$R = \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & -7 \\ 0 & -9 \end{bmatrix}$$

b)

$$U = \frac{1}{\sqrt{35}} \begin{bmatrix} 5 & 1+3i \\ 1-3i & -5 \end{bmatrix}$$

$$U^* A U = \begin{bmatrix} 3i & 119+42i \\ 0 & 105i \end{bmatrix}$$

c)

$$e^{At} = \frac{1}{3} \begin{bmatrix} \sin(3t) + 3\cos(3t) & -5\sin(3t) \\ 2\sin(3t) & -\sin(2t) + 3\cos(3t) \end{bmatrix}$$

Problem II: a)

$$\mathbf{x}'(t) = \begin{bmatrix} \sin t \\ 1 - \cos t \end{bmatrix}$$

b)

$$\mathbf{T}(t) = \frac{1}{\sqrt{2(1-\cos t)}} \begin{bmatrix} \sin t \\ 1 - \cos t \end{bmatrix} = \begin{bmatrix} \cos(t/2) \\ \sin(t/2) \end{bmatrix}$$

$$\mathbf{N}(t) = \begin{bmatrix} -\sin(t/2) \\ \cos(t/2) \end{bmatrix}$$

c)

$$L = \int_0^{2\pi} \sqrt{2(1-\cos t)}dt = \int_0^{2\pi} \sin(t/2)dt = 8$$

d)

$$\kappa(t) = \frac{1}{4\sin(t/2)}$$

Problem III: a)

$$\theta = \cos^{-1}(1/7) \ .$$

b)

$$\mathbf{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

c)

$$\sqrt{4/7}I + (1-\sqrt{4/7})\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} + \sqrt{3/7}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Problem IV: a)

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ \mu(1-x^2) - x \end{bmatrix}$$

b)

$$(\frac{1}{2\mu}(1+\sqrt{1+4\mu^2}), 0)\ ,\ (\frac{1}{2\mu}(1-\sqrt{1+4\mu^2}), 0)$$

c)

$$\begin{bmatrix} 0 & 1 \\ 2+\sqrt{1+4\mu^2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2-\sqrt{1+4\mu^2} & 0 \end{bmatrix}$$

e) The first critical point is always a saddle, and therefore unstable. The stability properties of the second depend on μ. If $\mu^2 < 3/4$ then the critical point is a center. If $\mu^2 < 3/4$ then the critical is an unstable saddle.