Test 1 for Calculus III for CS Majors, Math 2605 T1-T2, September 27, 2011

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: Consider the function

$$f(x, y) = x^3 + y^3 - 3x^2 y \ .$$

a) (5 points) Calculate the gradient at the point $(1, 1)$.

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 - 6xy \\ 3y^2 - 3x^2 \end{bmatrix}$$

Evaluated at $(1, 1)$ yields

$$\nabla f(1, 1) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

b) (10 points) Find the plane that is tangent to the graph of $f(x, y)$ at the point $(1, 1)$.

$$z = f(1, 1) + \nabla f(1, 2) \cdot (\mathbf{x} - \mathbf{x_0}) = -1 - 3(x - 1) \ ,$$

or

$$z + 3x = 2 \ .$$

c) (10 points) Find the points on the curve $f(x, y) = f(1, 1)$ where the tangent line is horizontal.

This means that the gradient is vertical and hence the x-component has to vanish. Hence we have to solve

$$3x^2 - 6xy = 0 \ , \ x^3 + y^3 - 3x^2 y = -1$$

From the first equation we get $x = 0$ or $2y = x$. The first case leads to $y^3 = -1$ and hence $y = -1$. Thus $(0, -1)$ is such a point. For the second alternative we get

$$8y^3 + y^3 - 12y^3 = -3y^3 = -1$$

and hence $y = \frac{1}{3^{1/3}}$ and $x = \frac{2}{3^{1/3}}$.

d) (10 points) Below is the plot of level curves of a function. Indicate in the picture below which of the arrows correspond to gradients of a function.

Problem 2: a) (10 points) Calculate the critical points of the function

$$f(x,y) = \frac{1}{3}x^3 + yx - y$$

$$\nabla f = \begin{bmatrix} x^2 + y \\ x - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

leads to $x = 1$ and $y = -1$.
b) (10 points) Calculate the Hessian at these critical points.

$$H_f(1,-1) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

c) (10 points) By computing the eigenvalues of the Hessian at these critical points, find the type of these critical points. Are they a max a min or a saddle?

The eigenvalues are $1+\sqrt{2}$ and $1-\sqrt{2}$ and hence the critical point is a saddle.

Problem 3: (10 points) a) Set up Newton's scheme for solving the equation $x^2 - y^2 - 1 = 0$ and $2xy - 1 = 0$.

$$\mathbf{x}_1 = \mathbf{x}_0 - J_F(\mathbf{x}_0)^{-1} F(\mathbf{x}_0)$$

$$F(\mathbf{x}) = \begin{bmatrix} x^2 - y^2 - 1 \\ 2xy - 1 \end{bmatrix}$$

$$J_F(\mathbf{x}) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

b) (15 points) Use as an initial guess the point $\mathbf{x_0} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and calculate the next approximation $\mathbf{x_1}$.

$$F(\mathbf{x}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$J_F(\mathbf{x}) = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J_F(\mathbf{x})^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$$

c) (5 points) Check whether this leads to an improvement.

$$|F(\mathbf{x}_0)| = 1$$

$$|F(\mathbf{x}_1)| = \frac{\sqrt{97}}{4}$$

which is not an improvement.

Problem 4: Consider the function $f(x,y) = x^2 - y^2 + \frac{1}{\sqrt{2}}xy$ on the unit disk $\{(x,y) : x^2 + y^2 \leq 1\}$.

a) (5 points) Find the critical points of this function in the interior of the unit disk, i.e., in $\{(x,y) : x^2 + y^2 < 1\}$.

$$\nabla f(x,y) = \begin{bmatrix} 2x + \frac{1}{\sqrt{2}}y \\ -2y + \frac{1}{\sqrt{2}}x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the only solution is $(0,0)$. In other words the origin is the only critical point.

b) **Extra credit** (15 points) Using Lagrange multipliers, find the maximum and minimum of $f(x,y)$ on the boundary of the disk, i.e., on $\{(x,y) : x^2 + y^2 = 1\}$. Find the maximum and minmum of $f(x,y)$ on $\{(x,y) : x^2 + y^2 \leq 1\}$.

The equations for the Lagrange method are

$$\begin{bmatrix} 2x + \frac{1}{\sqrt{2}}y \\ -2y + \frac{1}{\sqrt{2}}x \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ y \end{bmatrix} \ , \ x^2 + y^2 - 1 = 0 \ .$$

Cross-multiplication leads to

$$(2x + \frac{1}{\sqrt{2}}y)y = (-2y + \frac{1}{\sqrt{2}}x)x$$

or

$$(x^2 - y^2) - 4\sqrt{2}xy = 0 \ .$$

This can be written as

$$(x - 2\sqrt{2}y)^2 = 9y^2$$

or

$$x - 2\sqrt{2}y = \pm 3y \ .$$

The first possibility is

$$x = (3 + 2\sqrt{2})y$$

and plugging this into the equation $x^2 + y^2 = 1$ yields Case 1:

$$y = \pm \frac{1}{\sqrt{(3+2\sqrt{2})^2 + 1}} = \pm \frac{1}{\sqrt{18 + 12\sqrt{2}}} \ , \ x = \pm \frac{3 + 2\sqrt{2}}{\sqrt{18 + 12\sqrt{2}}} \ .$$

The values of the function at these two points are both equals to

$$\frac{12 + 9\sqrt{2}}{12 + 8\sqrt{2}} > 0 \ .$$

The other solutions is

$$x = (2\sqrt{2} - 3)y$$

which leads to Case 2:

$$y = \pm\frac{1}{\sqrt{18-12\sqrt{2}}}\ ,\ \ x = \pm\frac{2\sqrt{2}-3}{\sqrt{18-12\sqrt{2}}}$$

The values of the function at these points is

$$\frac{36-27\sqrt{2}}{36-24\sqrt{2}} < 0\ .$$

Since the function vanishes at the only critical point, we have that the function has a maximum in Case 1 and a minimum in Case 2.