Test 2 for Calculus III for CS Majors, Math 2506 T1-T2, October 25, 2011

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: Consider the function

$$f(x,y) = \frac{1}{1+x^2+(y-2)^2} \ .$$

a) (10 points) Find the critical points of $f(x,y)$ inside the domain $x^2+y^2 < 1$, if there are any, and determine their type, i.e., a local minimum, local maximum or saddle.

$$\nabla f(x,y) = -\frac{1}{(1+x^2+(y-2)^2)^2}\begin{bmatrix} 2x \\ 2(y-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence the unique critical point has the coordinates $(0,2)$ which is not inside the unit circle.
b) (10 points) Find the maximum and minimum of the function $f(x,y)$ on the boundary, i.e., subject to the constraint $x^2+y^2-1=0$.

The Lagrange multiplier technique leads to

$$-\frac{1}{(1+x^2+(y-2)^2)^2}\begin{bmatrix} 2x \\ 2(y-2) \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

with $x^2+y^2-1=0$. These are two equations. Multiplying the first equation with y, the second with x and subtracting, thus eliminating λ, leads to

$$-\frac{1}{(1+x^2+(y-2)^2)^2}xy = -\frac{1}{(1+x^2+(y-2)^2)^2}(y-2)x \ ,$$

or

$$xy = (y-2)x$$

which yields $x=0$. Hence $y=\pm 1$. At the point $(0,1)$ the value of the function is $1/2$ and at $(0,-1)$ the value is $1/10$.
c) (5 points) Find the maximum and minimum of the function $f(x,y)$ in the region $x^2+y^2 \leq 1$.

Since there is no critical point inside the circle, the maximum and minimum must be attained on the boundary. Hence the minimum value of the function on the whole unit disk is $1/10$ attained at the point $(0,-1)$ and its maximum value is $1/2$ attained at the point $(0,1)$.

Problem 2: Consider the matrix

$$A = \begin{bmatrix} 10 & 0 & 3 \\ 0 & 5 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

a) (5 points) Compute Off(A).

$$Off(A) = 20 \ .$$

b) (10 points) Calculate the Givens matrix G for the first step in the Jacobi iteration for diagonalizing A by picking the 2×2 submatrix with the largest off diagonal elements. (You do not have to calculate $G^T AG$.)

Pick the submatrix

$$\begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}$$

which has the eigenvalues 1 and 11 with the corresponding eigenvectors (in the same order)

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix} , \ \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Hence the corresponding Givens matrix is

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & \sqrt{10} & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

Note that this procedure is not unique. You can interchange the two eigenvalues and the eigenvectors. You would get a different Givens matrix, but nevertheless it removes the off-diagonal 3.

c) (10 points) Calculate Off$(G^T AG)$.

$$\text{Off}(A^{(1)}) = \text{Off}(A) - 2 \times 3^2 = 20 - 18 = 2 \ .$$

Problem 3: (10 points) a) Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \\ 4 & 2 \end{bmatrix} .$$

The singular value decomposition is given by

$$V = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} , \ U = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} , \ D = 3\sqrt{5} .$$

Assume that

$$V = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -2 & 2 \\ 2 & 1 \end{bmatrix} , D = \begin{bmatrix} 6\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} , U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

form the singular decomposition of a matrix B.
b) (10 points) Calculate the generalized inverse B^+

$$B^+ = UD^{-1}V^T = \frac{1}{36}\begin{bmatrix} -3 & -6 & 0 \\ 5 & 2 & 4 \end{bmatrix}$$

c) (5 points) Find the least square solution of the problem $Bx = b$ where

$$b = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$x = B^+ b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Problem 4: a) (10 points) Diagonalize the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$$

-3 is an eigenvalue with eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and 2 is an eigenvalue with eigenvector $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

Hence,

$$A = VDV^{-1}$$

where

$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}, \quad V^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}.$$

b) (10 points) Find a Schur factorization of the matrix A.

Pick the eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, normalize and complement it to an orthogonal matrix

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Next compute

$$T = U^T A U = \begin{bmatrix} -3 & 3 \\ 0 & 2 \end{bmatrix}$$

Thus $A = UTU^T$ with U and T as above. The Schur factorization is not unique. We could have used the other eigenvector instead.

c) (5 points) What is the Schur factorization of A^2?

$$A^2 = UT^2U^T$$

with U as above and

$$T^2 = \begin{bmatrix} 9 & -3 \\ 0 & 4 \end{bmatrix}.$$

Extra Credit: (10 points) Find the Householder reflection that maps the vector $\mathbf{x}$ to the vector $\mathbf{y}$ where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}.$$

$$\mathbf{u} = \frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then the Householder reflection is given by

$$I - 2\mathbf{u}\mathbf{u}^T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$