Test 3 for Calculus III for CS Majors, Math 2506 T1-T2, November 22, 2011

Name:

This test is to be taken without calculators and notes of any sort. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. State your work clearly, otherwise credit cannot be given.

Problem 1: a) (10 points) Find the Q-R factorization of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ -4 & -7 \end{bmatrix} .$$

$$\mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$Q = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$$

and

$$QA = \begin{bmatrix} 5 & 8 \\ 0 & 1 \end{bmatrix} =: R$$

Of course

$$A = QR .$$

b) (10 points) Find the Schur factorization of the matrix A.

A has eigenvalues 1 and -5. The normalized eigenvector associated with the eigenvalue 1 is

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} .$$

Hence we set

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} ,$$

and

$$T = U^T A U = \begin{bmatrix} 1 & -8 \\ 0 & -5 \end{bmatrix} .$$

Of course

$$A = U^T T U .$$

Problem 2: a) (10 points) Find the rotation Q_0 that has an angle of rotation of π and rotation axis

$$\frac{1}{\sqrt{17}}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}.$$

Euler's formula reads

$$\cos\theta I + (1-\cos\theta)\mathbf{e}\mathbf{e}^T + \sin\theta B_{\mathbf{e}}$$

where

$$B_{\mathbf{e}} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix}$$

In our example $\cos\pi = -1$ and $\sin\pi = 0$. Hence

$$Q = -I + 2\mathbf{e}\mathbf{e}^T = -I + \frac{2}{17}\begin{bmatrix} 9 & 6 & 6 \\ 6 & 4 & 4 \\ 6 & 4 & 4 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1 & 12 & 12 \\ 12 & -9 & 8 \\ 12 & 8 & -9 \end{bmatrix}.$$

b) (10 points) Find the angle and axis of rotation for the matrix

$$Q_1 = \frac{1}{9}\begin{bmatrix} 1 & 8 & 4 \\ 4 & -4 & 7 \\ 8 & 1 & -4 \end{bmatrix}$$

$$\cos\theta = \frac{\mathrm{Tr}(Q_1) - 1}{2} = -\frac{8}{9}$$

Further

$$Q - Q^T = \frac{1}{9}\begin{bmatrix} 0 & 4 & -4 \\ -4 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix}.$$

Thus

$$\mathbf{e} = \frac{1}{\sqrt{17}}\begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix}$$

c) (5 points) What do Q_0 and Q_1have in common?

Their axes of rotation are parallel.

Problem 3: Consider the differential equation

$$x'' = x' + x - x^2 \ .$$

a) (5 points) Write this equation as a first order system of equations.

$$\mathbf{x}' = \begin{bmatrix} y \\ y + x - x^2 \end{bmatrix} \ .$$

b) (5 points) Find the equilibrium points.

$$(0,0) \ , \ (1,0) \ .$$

c) (10 points) Linearize the system at these equilibrium points.

Around the equilibrium point $(0,0)$ we have

$$\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{z}$$

and around $(1,0)$

$$\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{z}$$

d) (10 points) For these linear systems determine the type and stability of these equilibrium points.

The point $(0,0$ is a saddle since the determinant equals -1. This point remains a saddle also for the non-linear system.

The linearized system at the point $(1,0)$ has the characteristic polynomial $\lambda^2 - \lambda + 1$ and hence the eigenvalues

$$\lambda_\pm = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

which shows that the motion in the vicinity of the critical point is an unstable spiral.

Problem 4: Consider the following initial value problem

$$\begin{aligned} x' &= 5x + 4y \\ y' &= 4x - y \end{aligned} \;, \; x(0) = 1 \;,\; y(0) = 3 \;.$$

a) (5 points) Bring the above system into the form $\mathbf{x}' = A\mathbf{x}$ where A is a 2×2 matrix.

$$\mathbf{x}' = A\mathbf{x} \;,\; \mathbf{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

where

$$\begin{bmatrix} 5 & 4 \\ 4 & -1 \end{bmatrix} .$$

b) (10 points) Find the eigenvalues and eigenvectors of A.

$$\lambda^2 - 4\lambda - 21 = 0$$

$$\lambda_1 = 7 \;, \lambda_2 = -3 \;.$$

The corresponding eigenvalues are

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$

c) (10 points) Solve the initial value problem.
The general solutions is given by

$$ae^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + be^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and with $a = b = 1$ the initial conditions are satisfied. Hence

$$e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

is the desired solution.