Quiz 3 for Calculus ++, Math 2605 T1-2, October 13, 2011

Name:

This quiz is to be taken without calculators and notes of any sorts. The allowed time is 20 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

I: (3 points) Find the maximum and minimum of the function $f(x,y) = 2x + y$ on the circle $x^2 + y^2 = 1$. Use Lagrange multipliers.

With $g(x,y) = x^2 + y^2 - 1$ we have

$$\nabla f = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \nabla g = 2 \begin{bmatrix} x \\ y \end{bmatrix} .$$

At the extremal points the two gradients have to be parallel and so

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

which, by cross-multiplication leads to $2y = x$. This together with $g = 0$ yields $5y^2 = 1$ and hence $y = \pm 1/\sqrt{5}$. Thus the points are

$$\vec{x}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{x}_2 = -\frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} .$$

$f(\vec{x}_1) = \sqrt{5}$ is the max and $f(\vec{x}_2) = -\sqrt{5}$ is the min.

II: a) (2 points) Find the eigenvalues of the matrix $\begin{bmatrix} 1 & 3 \\ 3 & -7 \end{bmatrix}$

The characteristic polynomial is given by

$$\lambda^2 + 6\lambda - 16 = (\lambda + 3)^2 - 25 = 0$$

which has the roots $2, -8$ and these are the two eigenvalues.

b) (2 points) The matrix $\begin{bmatrix} 3 & 3 \\ 3 & -5 \end{bmatrix}$ has the eigenvalues 4 and -6. Find the normalized eigenvectors.

Solving the equation $3x+3y = 4x$ yields $x = 3c, y = c$ where c is an arbitrary constant. Hence

$$\frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

is the normalized eigenvector associated with the eigenvalue 4.

The other eigenvector is orthogonal and is given by

$$\frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} .$$

III: (3 points) The matrix $\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}$ has the normalized eigenvectors $\frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$
Calculate the Givens matrix in the first step of the Jacobi algorithm for diagonalizing the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 4 & 0 & 3 \\ 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & -4 \end{bmatrix}$$

The Givens matrix is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{10}} & 0 & \frac{-1}{\sqrt{10}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{10}} & 0 & \frac{3}{\sqrt{10}} \end{bmatrix}$$