Final Exam for Calculus II, Math 1502, December 15, 2010
Name:

## Section:

Name of TA:
This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414 \ldots$. Show your work, otherwise credit cannot be given. Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

| Problem | Score |
| :--- | :--- |
| I |  |
| II |  |
| III |  |
| IV |  |
| V |  |
| VI |  |
| VII |  |
| VIII |  |
| IX |  |
| X |  |
| XI |  |
| XII |  |
| Total |  |

## Name:

## Section:

Name of TA:
Problems related to Block 1:
I: (15 points) Compute with an error less than $10^{-3}$

$$
\begin{gathered}
I=\int_{0}^{1} e^{x^{4}} d x \\
\left|I-\sum_{k=0}^{n} \frac{1}{k!} \frac{1}{4 k+1}\right| \leq \frac{3}{(n+1)!(4 n+5)}
\end{gathered}
$$

This is less than $10^{-3}$ for $n=5$.
II: a) (7 points) Compute the limit

$$
\lim _{x \rightarrow 0} \frac{\log (1+x)+(1-x)-\cos x}{x^{3}}
$$

By l'Hopital or Taylor: Answer $\frac{1}{3}$.
b) (8 points) Does the improper integral

$$
\int_{0}^{1} \frac{1}{x^{2}} e^{-\frac{1}{x}} d x
$$

exist? If yes, compute it.

$$
\lim _{\varepsilon \rightarrow 0} \int_{\varepsilon}^{1} \frac{1}{x^{2}} e^{-\frac{1}{x}} d x=\int_{1}^{\frac{1}{\varepsilon}} e^{-u} d u=e^{-1}-e^{-\frac{1}{\varepsilon}}
$$

which converges to $e^{-1}$.

## Name:

## Section:

Name of TA:

Problems related to Block 2:
III: a) (7 points) Is the series

$$
\sum_{k=2}^{\infty} \frac{k^{\log k}}{(\log k)^{k}}
$$

convergent?
Root test:

$$
\lim _{k \rightarrow \infty} \frac{k^{\frac{\log k}{k}}}{\log k}=0
$$

since

$$
k^{\frac{\log k}{k}} \rightarrow 1
$$

and $\log k \rightarrow \infty$. Note that

$$
\log \left(k^{\frac{\log k}{k}}\right)=\frac{(\log k)^{2}}{k} \rightarrow 0
$$

as $k \rightarrow \infty$.
b) (8 points) Find the interval of convergence of the power series

$$
\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} x^{k}
$$

Power series converges absolutely for $|x|<1$. At $x=1$

$$
\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}
$$

does not converge. At $x=-1$ we have

$$
\sum_{k=1}^{\infty}(-10)^{k} \frac{1}{\sqrt{k}} x^{k}
$$

and since $1 / k$ decreases to zero, we have convergence. Thus the interval of convergence is $[-1,1)$.

IV: (15 points) Solve the initial value problem

$$
\begin{gathered}
y^{\prime}+(x+1) y=e^{-\frac{x^{2}}{2}-x}, y(0)=1 \\
\left(y e^{\frac{x^{2}}{2}-x}\right)^{\prime}=1 \\
y(x)=(x+c) e^{-\frac{x^{2}}{2}-x}
\end{gathered}
$$

initial condition yield $c=1$.

$$
y(x)=(x+1) e^{-\frac{x^{2}}{2}-x}
$$

## Name:

## Section:

Name of TA:
Problems related to Block 3:
$\mathbf{V}$ : (20 points) Find a one-one parametrization for the solution of the systems below provided the solution exists:

$$
\begin{gathered}
x+2 y-6 z=2 \\
2 x+y+3 z=1 \\
3 x+y+7 z=1
\end{gathered}
$$

$$
\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]+t\left[\begin{array}{c}
-4 \\
5 \\
1
\end{array}\right]
$$

$$
\begin{gathered}
x+2 y+7 z=1 \\
x+3 y+12 z=1 \\
-x+4 y+23 z=1
\end{gathered}
$$

No solution.

$$
\left[\begin{array}{lll|l}
1 & 2 & 7 & 1 \\
0 & 1 & 5 & 0 \\
0 & 0 & 0 & 2
\end{array}\right]
$$

VI: (15 points) A plane in $\mathbb{R}^{3}$ has the parametric representation $\vec{x}(s, t)=\vec{x}_{0}+s \vec{v}_{1}+t \vec{v}_{2}$ where $\vec{x}_{0}=\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right], \vec{v}_{1}=\left[\begin{array}{l}2 \\ 1 \\ 3\end{array}\right], \vec{v}_{2}=\left[\begin{array}{l}3 \\ 2 \\ 1\end{array}\right]$. Find an equation for this plane. $-5 x+7 y+z=3$.

## Name:

## Section:

## Name of TA:

VII: (20 points) Consider the matrix

$$
A=\left[\begin{array}{lll}
1 & 2 & 4 \\
2 & 3 & 5 \\
3 & 4 & 6
\end{array}\right]
$$

a) Find a basis for $\operatorname{Ker}(A)$ and a basis for $\operatorname{Img}(A)$,

Row reduction leads to

$$
\left[\begin{array}{lll}
1 & 2 & 4 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{array}\right]
$$

Thus, the vectors

$$
\left[\begin{array}{l}
1 \\
2 \\
3
\end{array}\right],\left[\begin{array}{l}
2 \\
3 \\
4
\end{array}\right]
$$

form a basis for $\operatorname{Img}(A)$ and the vector

$$
\left[\begin{array}{c}
2 \\
-3 \\
1
\end{array}\right]
$$

is a basis for $\operatorname{Ker}(A)$.
b) Find a basis for $\operatorname{Ker}\left(A^{T}\right)$ and $\operatorname{Img}\left(A^{T}\right)$.

$$
\operatorname{Ker}\left(A^{T}\right)=\operatorname{Img}(A)^{\perp}
$$

and hence the vector

$$
\left[\begin{array}{c}
1 \\
-2 \\
1
\end{array}\right]
$$

is a basis for $\operatorname{Ker}\left(A^{T}\right)$.

$$
\operatorname{Img}\left(A^{T}\right)=\operatorname{Ker}(A)^{\perp}
$$

and a basis is given by

$$
\left[\begin{array}{l}
3 \\
2 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
3
\end{array}\right]
$$

which of course is not unique.
VIII: (15 points) Find the QR factorization of the matrix

$$
\begin{gathered}
{\left[\begin{array}{ccc}
1 & 6 & 4 \\
2 & 3 & -1 \\
2 & -6 & -10
\end{array}\right]} \\
Q=\frac{1}{3}\left[\begin{array}{cc}
1 & 2 \\
2 & 1 \\
2 & -2
\end{array}\right] \\
R=Q^{T} A=\left[\begin{array}{ccc}
3 & 0 & -6 \\
0 & 9 & 9
\end{array}\right]
\end{gathered}
$$

## Name:

## Section:

## Name of TA:

IX: (15 points) Consider the curve given by the equation

$$
8 x^{2}+6 x y=1
$$

a) What is the type of the curve? Is it an ellipse or hyperbola?

Associated matrix

$$
\left[\begin{array}{ll}
8 & 3 \\
3 & 0
\end{array}\right]
$$

Determinant is negative and hence hyperbola.
b) Graph the curve in a qualitative fashion below. Indicate in the figure the directions of the eigenvectors.
Eigenvalues: $\mu^{2}-8 \mu-9=(\mu-9)(\mu+1)=0$

$$
\mu_{1}=9, \mu_{2}=-1
$$

Eigenvectors

$$
\begin{aligned}
& \vec{u}_{1}=\frac{1}{\sqrt{10}}\left[\begin{array}{l}
3 \\
1
\end{array}\right] \\
& \vec{u}_{2}=\frac{1}{\sqrt{10}}\left[\begin{array}{c}
-1 \\
3
\end{array}\right]
\end{aligned}
$$

$\mathbf{X}:(15$ points $)$ Find the eigenvalues and eigenvectors of the following matrices. What is their algebraic multiplicity, what is their geometric multiplicity?
a) $\left[\begin{array}{lll}1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1\end{array}\right]$
b) $\left[\begin{array}{ll}2 & 9 \\ 4 & 7\end{array}\right]$
a) $\mu_{1}=1$ has algebraic multiplicity 3 and geometric multiplicity 1 . The vector

$$
\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]
$$

is the only eigenvector.
b) Eigenvalues: $\mu^{2}-9 \mu-22=(\mu-11)(\mu+2)=0$

$$
\mu_{1}=11, \mu_{2}=-2 .
$$

The algebraic multiplicity and the geometric are the same in both cases. Eigenvectors:

$$
\vec{u}_{1}=\left[\begin{array}{l}
1 \\
1
\end{array}\right], \vec{u}_{2}=\left[\begin{array}{c}
9 \\
-4
\end{array}\right]
$$

## Name:

## Section:

## Name of TA:

XI: (20 points) Using the superposition principle, solve the initial value problem given by the system

$$
\begin{gathered}
x^{\prime}=2 x+9 y \\
y^{\prime}=4 x+7 y \\
x(0)=1, y(0)=2
\end{gathered}
$$

Same problem as before: The general solution is given by

$$
a e^{11 t}\left[\begin{array}{l}
1 \\
1
\end{array}\right]+b e^{-2 t}\left[\begin{array}{c}
9 \\
-4
\end{array}\right]
$$

For the initial condition we solve

$$
\begin{gathered}
a\left[\begin{array}{l}
1 \\
1
\end{array}\right]+b\left[\begin{array}{c}
9 \\
-4
\end{array}\right]=\left[\begin{array}{l}
1 \\
2
\end{array}\right] \\
a=\frac{22}{13}, b=-\frac{1}{13}
\end{gathered}
$$

XII: ( 15 points) Solve the recursive relation, i.e., find $a_{n}$ for arbitrary values of $n$,

$$
a_{n+1}=5 a_{n}-4 a_{n-1}
$$

with $a_{0}=1, a_{1}=2$.

$$
\vec{x}_{n}=\left[\begin{array}{c}
a_{n} \\
a_{n-1}
\end{array}\right]
$$

then

$$
\vec{x}_{n+1}=A \vec{x}_{n}
$$

where

$$
A=\left[\begin{array}{cc}
5 & -4 \\
1 & 0
\end{array}\right]
$$

Hence

$$
\vec{x}_{n}=A^{n-1} \vec{x}_{1}, \vec{x}_{1}=\left[\begin{array}{l}
2 \\
1
\end{array}\right]
$$

Eigenvalues: $\mu^{2}-5 \mu+4=(\mu-4)(\mu-1)=0$

$$
\mu_{1}=4, \mu_{2}=1
$$

and the corresponding eigenvectors are

$$
\vec{u}_{1}=\left[\begin{array}{l}
4 \\
1
\end{array}\right], \vec{u}_{2}=\left[\begin{array}{l}
1 \\
1
\end{array}\right]
$$

Hence

$$
A=\left[\begin{array}{ll}
4 & 1 \\
1 & 1
\end{array}\right]\left[\begin{array}{ll}
4 & 0 \\
0 & 1
\end{array}\right] \frac{1}{3}\left[\begin{array}{ll}
1 & 1 \\
1 & 4
\end{array}\right]
$$

and

$$
A^{n-1}=\left[\begin{array}{ll}
4 & 1 \\
1 & 1
\end{array}\right]\left[\begin{array}{cc}
4^{n-1} & 0 \\
0 & 1
\end{array}\right] \frac{1}{3}\left[\begin{array}{cc}
1 & -1 \\
-1 & 4
\end{array}\right]=\frac{1}{3}\left[\begin{array}{cc}
4^{n}-1 & 4-4^{n} \\
4^{n-1}-1 & 4-4^{n-1}
\end{array}\right]
$$

Together with the initial condition we get

$$
a_{n}=\frac{1}{3}\left[4^{n}+2\right] .
$$

