Test I for Calculus II, Math 1502 H1-H5 , September 11, 2012

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: Consider the function $f(x) = (8+x)^{1/3}$.
a) (10 points) Find the first order Taylor polynomial $P_1(x)$ (i.e., at $a = 0$) for $f(x)$ and the corresponding remainder in Lagrange form.

The first, second and third derivatives are given by

$$\frac{1}{3}(8+x)^{-2/3} \ , \ -\frac{2}{3^2}(8+x)^{-5/3} \ .$$

Taylor's theorem then reads

$$(8+x)^{1/3} = (8)^{1/3} + \frac{1}{3}(8)^{-2/3}x - \frac{1}{2!}\frac{2}{3^2}(8+c(x))^{-5/3}x^2$$

or

$$(8+x)^{1/3} = 2 + \frac{1}{3\cdot 2^2}x - \frac{1}{2!}\frac{2}{3^2}(8+c(x))^{-5/3}x^2$$

where $c(x)$ is between 0 and x.
b) (7 points) Using the above result compute an approximate value, call it A, for $9^{1/3}$.

Setting $x = 1$ in the above formula yields

$$(9)^{1/3} = 2 + \frac{1}{3\cdot 2^2} - \frac{1}{2!}\frac{2}{3^2}(8+c)^{-5/3}$$

with $0 < c < 1$. The approximate value is then given by

$$A = 2 + \frac{1}{3\cdot 2^2} \ .$$

c) (8 points) Give an estimate on how accurate the value computed in b) approximates $9^{1/3}$, i.e., give a bound on

$$|9^{1/3} - A| \ .$$

We have to bound the remainder by a number that is independent of c since we do not know what c is. It is certainly true that

$$\frac{1}{2!}\frac{2}{3^2}(8+c)^{-5/3} \le \frac{1}{2!}\frac{2}{3^2}(8)^{-5/3} = \frac{1}{2!}\frac{2}{3^2\cdot 2^5}$$

and hence

$$|9^{1/3} - A| \le \frac{1}{2!}\frac{2}{3^2\cdot 2^5} \ .$$

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II: a) (8 points) Compute

$$\lim_{x\to 0} \frac{e^x - 1}{\ln(1+x)} = \lim_{x\to 0} \frac{e^x}{(\frac{1}{1+x})} = 1 \ .$$

b) (9 points)

$$\lim_{x\to\infty} \frac{\int_x^\infty e^{-y^2}\,dy}{e^{-x^2}} = \lim_{x\to\infty} \frac{e^{-x^2}}{2xe^{-x^2}} = 0 \ .$$

c) (8 points)

$$\lim_{x\to\infty} x\sin(\frac{1}{x}) = \lim_{x\to 0} \frac{\sin x}{x} = 1 \ .$$

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III: Do the following two improper integrals exist? Make an educated guess first. You will get 3 points if your guess is correct. Then use the comparison theorem to prove your guess.

a) (7 points)

$$\int_0^\infty \frac{1}{\sqrt{1+x^2}}dx$$

Since the integrand behaves like $\frac{1}{x}$ for large values of x and $\int_1^\infty \frac{1}{x}dx$ does not exist we have to bound the integrand from *below* by a function whose improper integral does not exist and then use the comparison theorem. Splitting the integral

$$\int_0^\infty \frac{1}{\sqrt{1+x^2}}dx = \int_0^1 \frac{1}{\sqrt{1+x^2}}dx + \int_1^\infty \frac{1}{\sqrt{1+x^2}}dx$$

we have to concentrate on the second term since the first is OK. In the second term $x \geq 1$ and hence we have that

$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}}\frac{1}{x}$$

whose improper integral does not exist.

b) (8 points)

$$\int_1^\infty [\sin(1/x)]^2 dx$$

As x gets large, $1/x$ gets small and $\sin\frac{1}{x}$ is well approximated by $1/x$. Since

$$\int_1^\infty \frac{1}{x^2}dx$$

exists, we suspect that $\int_1^\infty [\sin(1/x)]^2 dx$ also exists. Thus we have to find an *upper bound* on the integrand by a function whose improper integral exists and use the comparison theorem. Now $|\sin y| \leq |y|$ and hence

$$[\sin(1/x)]^2 \leq \frac{1}{x^2}$$

and the improper integral exists.
c) (10 points) Does the following integral exist?

$$\int_{1/2}^{2} \frac{1}{x(\ln x)^2} dx$$

The problem is that $\log 1 = 0$. Hence we have to remove the piece of the integral near the zero of the logarithm. Hence we have to decide whether

$$\lim_{\epsilon \to 0} \int_{1/2}^{1-\epsilon} \frac{1}{x(\ln x)^2} dx$$

and whether

$$\lim_{\epsilon \to 0} \int_{1+\epsilon}^{2} \frac{1}{x(\ln x)^2} dx$$

exists. Pick the first one and make the u substitution $u = \log x$ so that $du = \frac{dx}{x}$. Hence

$$\int_{1/2}^{1-\epsilon} \frac{1}{x(\ln x)^2} dx = \int_{\log \frac{1}{2}}^{\log(1-\epsilon)} \frac{1}{u^2} du = \frac{1}{\log \frac{1}{2}} - \frac{1}{\log(1-\epsilon)}$$

which diverges as ϵ approaches 0. Hence the integral does not exist. You could have used the other integral and compute

$$\int_{1+\epsilon}^{2} \frac{1}{x(\ln x)^2} dx = \int_{\log(1+\epsilon)}^{\log 2} \frac{1}{u^2} du = \frac{1}{\log(1+\epsilon)} - \frac{1}{\log 2}$$

which also diverges as ϵ tends to 0.

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IV: Which of the following series is convergent or divergent. If it is convergent, sum it.
a) (8 points)

$$\sum_{k=1}^{\infty}\left(\frac{k-1}{k}\right)^k .$$

$$\lim_{k\to\infty}\left(\frac{k-1}{k}\right)^k = \lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k = \frac{1}{e} \neq 0$$

Hence the series diverges.
b) (7 points)

$$\sum_{k=1}^{\infty}\log\frac{k+1}{k} .$$

$$\sum_{k=1}^{n}\log\frac{k+1}{k} = \sum_{k=1}^{n}[\log(k+1)-\log k] = \log(n+1)-\log 1 = \log(n+1)$$

and the series diverges.
c) The following series converges:

$$L = \sum_{k=0}^{\infty}\frac{3^k}{4^{k+1}}$$

Find L. (4 points) Find the smallest n so that $0 < L - s_n \leq \left(\frac{3}{4}\right)^{100}$. Here s_n is the n-th partial sum. (6 points)

$$L = \frac{1}{4}\sum_{k=0}^{\infty}\frac{3^k}{4^k} = \frac{1}{4}\sum_{k=0}^{\infty}\left(\frac{3}{4}\right)^k = \frac{1}{4}\frac{1}{1-\frac{3}{4}} = 1$$

Now

$$s_n = \frac{1}{4}\sum_{k=0}^{n}\frac{3^k}{4^k} = \frac{1}{4}\sum_{k=0}^{n}\left(\frac{3}{4}\right)^k = \frac{1}{4}\frac{1-\left(\frac{3}{4}\right)^{n+1}}{1-\frac{3}{4}} = 1-\left(\frac{3}{4}\right)^{n+1}$$

Hence

$$0 \leq L - s_n = \left(\frac{3}{4}\right)^{n+1} .$$

Hence the smallest n we are looking for is 99.