Test 2 for Calculus II, Math 1502 H1-H5, October 2, 2012

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414.... Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Section:

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I: Decide whether the following series converge or diverge. State which convergence test you are going to use.

a) (8 points)

$$\sum_{k=0}^{\infty} \frac{k^k}{2^{k^2}}$$

Root test: $\frac{k}{2^k}$ which tends to zero and hence the series converges. b) (8 points)

$$\sum_{k=1}^{\infty} \frac{k!}{(2k)!}$$

Ratio test: Have to compute $\frac{(k+1)!(2k)!}{(2(k+1))!k!} = \frac{(k+1)}{(2k+2)(2k+1)}$ which tends to zero. Hence the series converges.

c) (9 points)

$$\sum_{k=1}^{\infty} \frac{1}{k(\ln k)^2}$$

We use the integral test

$$\int_{2}^{\infty} \frac{1}{x(\ln x)^{2}} dx = \int_{\ln 2}^{\infty} \frac{1}{u^{2}} du$$

which exists. Hence the series converges.

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II: a) (9 points) Consider the alternating series

$$L = \sum_{k=0}^{\infty} (-1)^k 10^{-k}$$

Find the smallest value of N so that the N-th partial sum s_N satisfies $|L - s_N| < 10^{-15}$.

This is an alternating series and hence we can estimate $|L - s_N|$ by the next term in the series, 10^{-N-1} , and hence N = 15.

b) (8 points) Find the power series expansion for $\sinh x := \frac{1}{2}(e^x - e^{-x})$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and hence

$$\frac{1}{2}[e^x - e^{-x}] = \frac{1}{2} \left[\sum_{k=0}^{\infty} \frac{x^k}{k!} - \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} \right] = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!}$$

c) (8 points) Sum the series

$$\sum_{k=1}^{\infty} k 2^{-k}$$

(Hint: Differentiate the geometric series!)

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

By differentiation

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{x} \sum_{k=1}^{\infty} kx^k$$

Hence

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

and

$$\sum_{k=1}^{\infty} k 2^{-k} = \frac{1}{2} \frac{1}{(1-1/2)^2} = 2$$

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III: Find the interval of convergence of the following power series. State which convergence test you are going to use for computing the radius of convergence.

a) (8 points)

$$\sum_{k=0}^{\infty} \frac{1}{k^k} x^k$$

Apply the root test to $\sum_{k=0}^{\infty} \frac{1}{k^k} |x|^k$. We have to compute

$$\lim_{k \to \infty} \frac{|x|}{k} = 0$$

And hence the series converges for all x.

b) (9 points)

$$\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{x-2}{2} \right)^k$$

 Set

$$a_k = \frac{1}{k} \left(\frac{|x-2|}{2}\right)^k$$

and then use the ratio test:

$$\frac{a_{k+1}}{a_k} = \frac{k}{k+1} \frac{|x-2|}{2} \to \frac{|x-2|}{2}$$

Hence we have absolute convergence for all x with |x-2| < 2, i.e., 0 < x < 4. At x = 0 we have an alternating harmonic series which converges. At x = 4 it is the harmonic series which diverges. Hence the interval of convergence is $0 \le x < 4$.

c) (8 points)

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^k (x+1)^k$$

Ratio test:

$$\frac{\left(1+\frac{1}{(k+1)}\right)^{k+1}|x+1|^{k+1}}{\left(1+\frac{1}{k}\right)^{k}|x+1|^{k}} = \frac{\left(1+\frac{1}{(k+1)}\right)^{k+1}}{\left(1+\frac{1}{k}\right)^{k}}|x+1|^{k}$$

In the ratio above the numerator converges to e as well as the denominator. Hence the limit as $k \to \infty$ is |x+1|. thus we have absolute convergence for -2 < x < 0. At x = 0 we have the series

$$\sum_{k=1}^{\infty} \left(1 + \frac{1}{k} \right)^k$$

which diverges since $\left(1+\frac{1}{k}\right)^k \to e \neq 0$ as $k \to \infty$. At x = -2 the series is

$$\sum_{k=1}^{\infty} (-1)^k \left(1 + \frac{1}{k}\right)^k$$

which diverges for the same reason.

Section:

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IV: a) (12 points) Solve the initial value problem

$$y'' - y = 0$$
, $y(0) = 0$, $y'(0) = 1$.

The characteristic equation is $r^2 - 1 = 0$ with roots ± 1 . Hence the general solution is

$$y(x) = Ae^x + Be^{-x}$$

y(0) = 0 and hence A + B = 0. y'(0) = 1 and hence A - B = 1. Therefore we find that A = -B = 1/2 and

$$y(x) = \frac{e^x - e^{-x}}{2}$$

b) (13 points) At a certain moment, a tank contains 100 liters of brine with a concentration 40 grams of salt per liter. The brine is continuously drawn off at a rate of 20 liters per minute and replaced by brine containing 10 grams salt per liter. Find the amount of salt in the tank at time t later.

The volume is preserved at 100 liters since the same amount flows as flows in. If P(t) denotes the amount of salt in the tank we find

$$\frac{dP}{dt} = 200 - \frac{P(t)}{100}20 = 200 - \frac{P(t)}{5}$$

The integrating factor is $e^{t/5}$ and the solution is

$$P(t) = 1000 + Ce^{-t/5}$$

Since $P(0) = 100 \times 40 = 4000$, C = 3000 and

$$P(t) = 1000 + 3000e^{-t/5}$$