The Fourier transform, a summary

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is defined as the set of functions $f \in C^\infty(\mathbb{R}^d)$ with

$$\sup_{x \in \mathbb{R}^d} |x^\alpha D^\beta f(x)| \leq C_{\alpha,\beta} < \infty$$

for all multiindices $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$. $\beta_i, \alpha_i \geq 0$ integers. Here

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \quad \text{and}$$

$$D^\beta f(x) = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots \frac{\partial^{\beta_d}}{\partial x_d^{\beta_d}} f(x).$$

We shall denote $\sum_{i=1}^{d} \alpha_i = |\alpha|$.

For $f \in \mathcal{S}(\mathbb{R}^d)$ we define the <u>Fourier transform</u> of f by

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-2\pi i k \cdot x} f(x) dx$$

It is easy to see, using integration by parts, that

$$(-2\pi i k)^\beta \hat{f}(k) = \int_{\mathbb{R}^d} e^{-2\pi i k \cdot x} D^\beta f(x) dx \quad \text{and}$$

$$\left(\frac{1}{-2\pi i}\right)^{|\alpha|} (D^\alpha \hat{f})(k) = \int_{\mathbb{R}^d} e^{-2\pi i k \cdot x} x^\alpha f(x) dx$$

which implies that

$$|(k^\beta D^\alpha \hat{f})(k)| \leq 2\pi^{|\alpha|+|\beta|} \int |x^\alpha D^\beta f(x)|\,dx$$

$$\leq C_{\alpha,\beta} < \infty .$$

If we use operator notation

$$(\mathcal{F}f)(k) = \hat{f}(k)$$

we can say that $\mathcal{F}$ maps the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

It is an easy exercise to compute the Fourier transform of

$$g_\lambda(x) = e^{-\lambda \pi |x|^2} \in \mathcal{S}(\mathbb{R}^d)$$

and we get

$$\hat{g}_\lambda(k) = \lambda^{-\frac{d}{2}} e^{-\frac{\pi |k|^2}{\lambda}} \in \mathcal{S}(\mathbb{R}^d) .$$

Recall the inner product on $L^2(\mathbb{R}^d)$

$$(f,g) = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\,dx .$$

<u>Theorem 1</u> For $f, g \in \mathcal{S}(\mathbb{R}^d)$

$$(f,g) = (\mathcal{F}f, \mathcal{F}g).$$

<u>Proof:</u> We have that

$$\int_{\mathbb{R}^d} \overline{\hat{f}(k)}\, \hat{g}(k)\, e^{-\varepsilon\pi|k|^2} dk$$

$$= \int_{\mathbb{R}^{3d}} \overline{f(x)}\, g(y)\, e^{2\pi i(x-y)} e^{-\varepsilon\pi|k|^2} dx\, dy\, dk$$

$$= \int_{\mathbb{R}^{2d}} \varepsilon^{-\frac{d}{2}} e^{-\frac{\pi|x-y|^2}{\varepsilon}} \overline{f(x)}\, g(y)\, dx\, dy.$$

changing variables $y = x + \sqrt{\varepsilon} z$

$$= \int_{\mathbb{R}^{2d}} e^{-\pi|z|^2} \overline{f(x)}\, g(x+\sqrt{\varepsilon} z)\, dx\, dz$$

As $\varepsilon \to 0$ the first term tends to $(\hat{f}, \hat{g})$ whereas the last term to (f,g). $\blacksquare$

<u>Theorem 2</u> The Fourier transform maps $\mathcal{S}(\mathbb{R}^d)$ onto $\mathcal{S}(\mathbb{R}^d)$. In fact if $f \in \mathcal{S}(\mathbb{R}^d)$ then

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i k\cdot x} \hat{f}(k)\,dk.$$

<u>Proof:</u>

$$\int_{\mathbb{R}^d} \varepsilon^{-\frac{d}{2}} e^{-\frac{\pi|x-y|^2}{\varepsilon}} f(y)\,dy$$

$$= \int_{\mathbb{R}^d} e^{-\varepsilon\pi|k|^2} \hat{f}(k)\, e^{2\pi i k\cdot x}\,dk \text{ by Theorem 1.}$$

writing the first term, by changing variable $y = x + \sqrt{\varepsilon} z$, as

$$\int_{\mathbb{R}^d} e^{-\pi|z|^2} f(x+\sqrt{\varepsilon} z)\,dz$$

we find that as $\varepsilon \to 0$

$$f(x) = \int_{\mathbb{R}^d} \hat{f}(k)\, e^{2\pi i k\cdot x}\,dk.$$

$\blacksquare$

Thus we have the following situation:

$\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is invertible

and

$$\|\mathcal{F}f\|_2 = \|f\|_2.$$

Hence $\mathcal{F}$ can be uniquely extended to a unitary transformation on all of $L^2(\mathbb{R}^d)$.

Thus $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$

$$\|\mathcal{F}f\|_2 = \|f\|_2$$

and $\mathcal{F}$ is invertible.

$\|\mathcal{F}f\|_2 = \|f\|_2$ is called the Plancherel formula and

$(\mathcal{F}f, \mathcal{F}g) = (f, g)$ is called Parseval's identity.