1. About closed operators

In this summary we talk about unbounded operators. The situation is the following. We have a linear operator

$$A : D(A) \to \mathcal{H}$$

where $D(A)$ is a linear manifold, the **domain** of the operator A. An operator B is an **extension** of A if $D(A) \subset D(B)$ and $Af = Bf$ for all $f \in D(A)$. We write $A \subset B$.

Note that the domain is part of the definition of the operator. Consider, e.g., the linear operator $T_1 : C^1(\mathbb{R}) \to L^2(\mathbb{R})$ given by $T_1 f(x) = f'(x)$. The operator $T_2 : C_c^\infty(\mathbb{R}) \to \mathcal{H}$ is also given by $T_2 f(x) = f'(x)$. We treat these two operators as different operators. Clearly, $T_2 \subset T_1$. In general linear operators can be wild unless we impose some continuity.

Definition 1.1. Closed operators. *A linear operator $A : D(A) \to \mathcal{H}$ is* **closed** *if for any sequence of vectors $f_n \in D(A)$ such that, as $n \to \infty$, $f_n \to f$ and $Af_n \to g$, it follows that $f \in D(A)$ and $Af = g$.*

Bounded linear operators are obviously closed, in fact the convergence $f_n \to f$ entails the convergence of $Af_n \to Af$.

Another way of saying that an operator is closed is the following

Lemma 1.2. *A linear operator $A : D(A) \to \mathcal{H}$ is closed if and only if the domain $D(A)$ endowed with the norm $\|f\| + \|Af\|$ is a Banach space, i.e., a linear, normed, complete space.*

We have proved this lemma in class and it should not cause you any difficulty recovering the argument.

You may guess that neither the operator T_1 nor T_2 defined above is closed. This is a standard situation that occurs in practice. Rarely can we compute explicitly what a closed operator does to all its element in its domain. Thus, we have the definition

Definition 1.3. Closable operators *A linear operator $A : D(A) \to \mathcal{H}$ is* **closable** *if it has a closed extension.*

Here is a simple statement about closable operators.

Lemma 1.4. *A linear operator $A : D(A) \to \mathcal{H}$ is closable if and only if for any sequence $f_n \in D(A)$ such that, as $n \to \infty$, $f_n \to 0$ and $Af_n \to g$, it follows that $g = 0$.*

Proof. Assume that A is closable and denote by B a closed extension. If $f_n \in D(A)$ then $f_\in D(B)$. Since $Bf_n = Af_n \to g$ and since $f_n \to 0$, we have, since B is closed that $g = B0 = 0$. Conversely, consider the linear manifold

$$D = \{f \in \mathcal{H} : \text{ there exists } f_n \in D(A)\ , f_n \to f\ , Af_n \to g\}\ .$$

On D we define $\overline{A}f = g$. We have to show that g is independent of the sequence f_n. Let $u_n \in D(A)$ be another sequence with $u_n \to f$ and $Au_n \to h$. We have to show that $h = g$. Since $f_n - u_n \to 0$ and since $A(f_n - u_n) \to g - h$ it follows that $f = g$. Hence, $\overline{A}$ is defined on D and it is easy to see that it is a linear operator. It remains to show $\overline{A}$ is closed. Let f_n be a sequence in D such that $f_n \to f$ and $\overline{A}f_n \to g$. We have to show that $f \in D$ and $\overline{A}f = g$. Since for each n $f_n \in D$ there exists $u_n \in D(A)$ such that

$$\|f_n - u_n\| + \|\overline{A}f_n - Au_n\| < \frac{1}{n}\ .$$

Hence, $u_n \to f$ and $Au_n \to g$, i.e., $f \in D$ and $g = \overline{A}f$. Hence $\overline{A}$ is closed. □

We call $\overline{A}$ the **closure** of the (closable) operator A. It is the **smallest** closed extension of A in the sense that if $A \subset B$ and B is closed, then $\overline{A} \subset B$. We leave this as an easy exercise to the reader.

The notion of adjoint operators can easily be generalized to our new situation.

Definition 1.5. Adjoint operator *Let $A; D(A) \to \mathcal{H}$ be a linear operator (not necessarily closed) with $D(A) \subset \mathcal{H}$ dense. Define $D(A^*)$ to be the set of all elements $f \in \mathcal{H}$ such that the linear functional*

$$g \to (f, Ag)$$

extends to a bounded linear functional on all of $\mathcal{H}$. Since $D(A) \subset \mathcal{H}$ is dense, there exists, by the Riesz representation theorem a unique element $h \in \mathcal{H}$ such that

$$(f, Ag) = (h, g) \ .$$

We define $A^* f = h$. It is easily seen that A^* is a linear operator.

Here is another reason why the notion of closed operator makes sense.

Theorem 1.6. *Let A be a densely defined operator. Then the operator A^* is closed.*

Proof. Let $f_n \in D(A^*)$ such that $f_n \to f$ and $A^* f_n \to g$. Then for all $v \in D(A)$

$$(f, Av) = \lim_{n\to\infty} (f_n, Av) = \lim_{n\to\infty} (A^* f_n, v) = (g, v) \ .$$

Hence, $f \in D(A^*)$ and $g = A^* f$. □

Note, that we did not assume that A itself was closed, or even closable. The adjoint of any densely defined operator is automatically closed.

Here are a few simple facts.

Lemma 1.7. *a) If $A \subset B$ then $B^* \subset A^*$.*

b) If A is closable, then $(\overline{A})^* = A^*$.

Proof. $f \in D(B^*)$ means that there exists $h \in \mathcal{H}$ such that

$$(f, Bv) = (h, v)$$

for all $v \in D(B)$. Since $A \subset B$ we also have that

$$(f, Av) = (f, Bv) = (h, v)$$

for all $v \in D(A)$. Hence $B^* \subset A^*$. To prove b) note that since $A \subset \overline{A}$, $(\overline{A})^* \subset A^*$. Now let $f \in D(A^*)$. There exists a unique $h \in \mathcal{H}$ such that

$$(f, Av) = (h, v)$$

all $v \in D(A)$. If $w \in D(\overline{A})$ there exists $v_n \in D(A)$ such that $v_n \to w$ and $Av_n \to \overline{A}w$. Hence

$$(f, \overline{A}w) = \lim_{n\to\infty} (f, Av_n) = \lim_{n\to\infty} (h, v_n) = (h, w)$$

and hence $f \in D((\overline{A})^*)$. Since $h = A^* f$ we also have that $(\overline{A})^* f = h$. □

We have learned that by passing to adjoint operators one obtains closed operators. There is a natural closed extension for a closable operator A and that would be $(A^*)^*$. This operator, however, exists only if A^* is densely defined. The following theorem is a bit trickier than we have seen so far.

Theorem 1.8. *An linear operator $A : D(A) \to \mathcal{H}$ is closable if and only if A^* is densely defined, in which case $\overline{A} = (A^*)^*$.*

Proof. First we assume that A^* is densely defined. $f \in D((A^*)^*)$ means that there exists a unique $h = (A^*)^* f \in \mathcal{H}$ so that

$$(f, A^*u) = (h, u) = ((A^*)^* f, u)$$

for all $u \in D(A^*)$. If $f \in D(A)$, then for all $u \in D(A^*)$

$$(u, Af) = ((A^*u, f)$$

from which it follows that $f \in D((A^*)^*)$ and $(A^*)^* f = Af$. Hence $A \subset (A^*)^*$ and since $(A^*)^*$ is closed, the operator A is closable.

The proof of the converse is more difficult. Assume that A is closable. We have to show that $D(A^*)$ is dense. We want to show that the assumption that $D(A^*)$ is not dense leads to a contradiction. Since $(\overline{A})^* = A^*$ we may assume that $A = \overline{A}$, i.e., that A is closed. Since $D(A^*)$ is not dense, there exists a non-zero vector $f \in \mathcal{H}$ such that $f \perp D(A^*)$. Consider the minimization problem

$$D^2 = \inf_{g \in D(A)} \left[\|f - Ag\|^2 + \|g\|^2 \right] \ .$$

The idea for this expression is to approximate the pair $f, 0 \in \mathcal{H} \times \mathcal{H}$ by elements of the form $Ag, g \in \mathcal{H} \times D(A)$. The space $\mathcal{H} \times \mathcal{H}$ endowed with the inner product $(f_1, g_1) + (f_2, g_2)$ is a Hilbert space. Let $g_n \in D(A)$ be a minimizing sequence. As in the proof of the projection theorem one can show that g_n as well as Ag_n is a Cauhy sequence. Hence $g_n \to h$ for some $h \in \mathcal{H}$ and $Ag_n \to v \in \mathcal{H}$. Since A is closed $h \in D(A)$ and $Ah = v$. Thus, we have that

$$D^2 = \|f - Ah\|^2 + \|h\|^2 \ .$$

Pick any $v \in D(A)$ and consider

$$\|f - A(h + tv)\|^2 + \|(h + tv)\|^2 \geq D^2 \ .$$

As usual, this leads to the statement that

$$-(f - Ah, Av) + (h, v) = 0$$

for all $v \in D(A)$ or

$$(f - Ah, Av) = (h, v)$$

for all $v \in D(A)$. This means that $f - Ah \in D(A^*)$ and $A^*(f - Ah) = h$. Now, since $f \perp f - Ah$ we have that

$$\|f\|^2 = (f, Ah) \ ,$$

and in particular

$$\|f\| \leq \|Ah\| \ .$$

Further, since $h \in D(A)$

$$\|h\|^2 = (h, A^*(f - Ah) = (Ah, f) - \|Ah\|^2 = \|f\|^2 - \|Ah\|^2$$

or

$$\|h\|^2 + \|Ah\|^2 = \|f\|^2 \leq \|Ah\|^2 \ ,$$

which implies that $h = 0$ and hence $Ah = 0$, a contradiction, since $f \neq 0$. It remains to show that $\overline{A} = (A^*)^*$. We have seen before that, $\overline{A} \subset (A^*)^*$. To show the converse we may assume that A is closed and pick any $f \in D((A^*)^*)$. As before the problem

$$D^2 = \inf_{g \in D(A)} \left[\|f - g\|^2 + \|(A^*)^* f - Ag\|^2 \right]$$

has a minimizer $h \in D(A)$ (since A is closed). Again, we consider for $v \in D(A)$ arbitrary

$$\|f - (h + tv)\|^2 + \|(A^*)^* f - A(h + tv)\|^2 \geq D^2$$

and find

$$(f - h, v) + ((A^*)^* f - Ah, Av) = 0 \ .$$

This means that $(A^*)^* f - Ah \in D(A^*)$ and

$$A^*((A^*)^* f - Ah) = -(f - h)$$

Taking the inner product with $f - h$ yields

$$-\|f - h\|^2 = (A^*((A^*)^*(f - h), f - h) = (((A^*)^*(f - h), (A^*)^*(f - h)) \geq 0 \ .$$

Hence $f = g \in D(A)$ and $(A^*)^* f = Ag$. □

Our goal is to study a certain class of operators, the self-adjoint operators. We start with a defintion.

Definition 1.9. Symmetric operators *A linear operator $A : D(A) \to \mathcal{H}$ is symmetric if $D(A)$ is dense in $\mathcal{H}$ and for all $f, g \in D(A)$*

$$(Af, g) = (f, Ag) \ .$$

A simple consequence is that any symmetric operator A is extended by its adjoint, i.e.,

$$A \subset A^* \ ,$$

in other words **a symmetric operator is always closable**. If B is any symmetric extension of A,i.e., $A \subset B$ we have that $B^* \subset A^*$, i.e., we have

$$A \subset B \subset B^* \subset A^* \ .$$

A symmetric operator A with $A = A^*$ is called **self-adjoint**. Note that a self adjoint operator is automatically closed. Moreover, it does not have symmetric extensions. A symmetric operator A that has not symmetric extensions but $A \neq A^*$ is called maximally symmetric.

Here is a first simple criterion for slef-adjointess.

Theorem 1.10. *Let $A : D(A) \to \mathcal{H}$ be a symmetric operator with the property that $Ran(A) = \mathcal{H}$. Then A is selfadjoint.*

Proof. Since $D(A) \subset D(A^*)$ all we have to show is that $f \in D(A^*)$ implies $f \in D(A)$. Consider $g = A^* f$. Since $Ran(A) = \mathcal{H}$ there exists $h \in D(A)$ so that $g = Ah$. Now for all $v \in D(A)$

$$(f, Av) = (A^* f, v) = (g, v) = (Ah, v) = (h, Av) \ .$$

If $u \in \mathcal{H}$ is arbitrary, there exists $v \in D(A)$ such that $u = Av$. Hence we have for all $u \in \mathcal{H}$

$$(f, u) = (h, u)$$

and thus, $f = h \in D(A)$. □

The following surprising theorem is due to von Neumann.

Theorem 1.11. *Let $A : D(A) \to \mathcal{H}$ be a densely defined closed operator. Then, $D = \{h \in D(A) : Ah \in D(A^*)\}$ is dense and A^*A is a self-adjoint operator with domain D.*

Proof. We use a similar technique as before. For $f \in \mathcal{H}$ consider the minimization problem

$$D^2 = \inf_{g \in D(A)} \left[\|f - g\|^2 + \|Ag\|^2 \right] \ .$$

Let $g_n \in D(A)$ be a minimizing sequence, i.e.,

$$\|f - g_n\|^2 + \|Ag_n\|^2 \to D^2 \ .$$

Let us go through the argument in detail. By the parallelogram identity we find

$$\|\frac{(f - g_n) + (f - g_m)}{2}\|^2 + \|\frac{(f - g_n) - (f - g_m)}{2}\|^2 + \|\frac{Ag_n + Ag_m}{2}\|^2 + \|\frac{Ag_n - Ag_m}{2}\|^2$$
$$= \frac{1}{2}\|f - g_n\|^2 + \frac{1}{2}\|Ag_n\|^2 + \frac{1}{2}\|f - g_m\|^2 + \frac{1}{2}\|Ag_m\|^2 \ .$$

Put in another way

$$\|f - \frac{g_n + g_m}{2}\|^2 + \|\frac{A(g_n + g_m)}{2}\|^2 + \|\frac{g_n - g_m}{2}\|^2 + +\|\frac{Ag_n - Ag_m}{2}\|^2$$

converges as $n, m \to \infty$ to D^2. Since

$$D^2 \le \|f - \frac{g_n + g_m}{2}\|^2 + \|\frac{A(g_n + g_m)}{2}\|^2$$

we must have that g_n as well as Ag_n are Cauchy sequences and hence converge to h resp. v. Since A is closed, $h \in D(A)$ and $Ah = v$. Hence

$$D^2 = \|f - h\|^2 + \|Ah\|^2 \ .$$

The usual variational argument $h \to h + tv, v \in D(A)$ leads to

$$-(f - h, v) + (Ah, Av) = 0$$

all $v \in D(A)$. Since $D(A)$ is dense $Ah \in D(A^*)$ and

$$A^*Ah = f - h$$

or

$$A^*Ah + h = f \ .$$

This means that for any $f \in \mathcal{H}$ there exists $h \in D$ with $A^*Ah + h = f$. This means that the operator $A^*A + I$ is surjective. If $h_1, h_2 \in D$ and $A^*Ah_1 + h_1 = f = A^*Ah_2 + h_2$ it follows that

$$A^*A(h_2 - h_1) + (h_2 - h_1) = 0 \ .$$

Since $h_2 - h_1 \in D$ we have that

$$\|A(h_2 - h_1)\|^2 + \|h_2 - h_1\|^2 = 0$$

and hence $A^*A + I$ is injective on D. It remains to show that the operator A^*A is symmetric. The domain D is dense, for suppose that $f \perp D$ then

$$f = A^*Ah + h$$

for a unique $h \in D$. Hence

$$0 = (h, f) = (h, A^*Ah) + (h, h) = \|Ah\|^2 + \|h\|^2$$

which implies that h and hence $f = 0$. Finally, A^*A is symmetric on D since for $f, g \in D$

$$(f, A^*Ag) = (Af, Ag) = (A^*Af, g) \quad .$$

Hence, $A^*A + I$ is symmetric on D and its range is the whole Hilbert space and hence it is self-adjoint.

$\square$