Final Exam A for Calculus II, K1-K6, Math 1502, December 10, 2013

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This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given. **Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.**

Problem	Score
I	
II	
III	
IV	
V	
VI	
VII	
VIII	
IX	
X	
Total	

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I: a) (10 points) Write the following integral in terms of an alternating series.

$$\int_0^1 e^{-x^2} dx \ .$$

$$e^{-y} = \sum_{k=0}^{\infty} (-1)^k \frac{y^k}{k!}$$

so that

$$e^{-x^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{k!}$$

and by integrating

$$\int_0^1 e^{-x^2} dx = \sum_{k=0}^{\infty} (-1)^k \frac{\int_0^1 x^{2k} dx}{k!} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)k!}$$

b) (5 points) Determine the smallest n so that the partial sum s_n of the series obtained in part a) approximates the integral with an error less than 10^{-3}.

$$|\int_0^1 e^{-x^2} dx - s_n| < \frac{1}{(2n+3)(n+1)!}$$

$n = 3$ yields $\frac{1}{9 \cdot 24} > 10^{-3}$. $n = 4$ yields $\frac{1}{11 \cdot 120} < 10^{-3}$.

II: a) (7 points) Compute the eighth order Taylor polynomial (at $a = 0$) of the function

$$\cos(x^2)$$

$$1 - \frac{x^4}{2} + \frac{x^8}{24}$$

b) (8 points) For which which values of $\alpha > 0$ does the integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^{\alpha}} dx$$

exist? Compute it for these values of α.

$$\int_2^L \frac{1}{x(\ln x)^\alpha} dx = \int_{\ln 2}^{\ln L} \frac{1}{u^\alpha} du = \frac{1}{-\alpha+1} u^{-\alpha+1}|_{\ln 2}^{\ln L}$$

The limit as $L \to \infty$ exists if $\alpha > 1$ and the its value is

$$\frac{1}{\alpha-1}(\ln 2)^{1-\alpha} \ .$$

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III: a) (7 points) Compute the interval of convergence of the series

$$\sum_{k=1}^{\infty} \frac{2^k}{k}(x-1)^k \ .$$

State the convergence test you are using.

b) (8 points) Does the series

$$\sum_{n=1}^{\infty} \frac{n^n}{3^{n \ln n}}$$

converge? State the convergence test you are using.

IV: (15 points) Find the solution of the initial value problem

$$\frac{dx}{dt} = \frac{x^2}{t} \ , \ x(1) = 1 \ .$$

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V: Consider the three vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}.$$

a) (8 points) Determine all the values for a and b for which the vector

$$\vec{b} = \begin{bmatrix} b \\ 2 \\ -1 \end{bmatrix}$$

is in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Row reduction of the matrix

$$\begin{bmatrix} 2 & 1 & 1 & b \\ 1 & 1 & -2 & 2 \\ 1 & -1 & a & -1 \end{bmatrix}$$

leads to

$$\begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & -1 & 5 & b-4 \\ 0 & 0 & a-8 & 5-2b \end{bmatrix}$$

All we need is that the system is consistent, i.e., $a \neq 8$ b arbitrary and $a = 8, b = 5/2$.

b) (7 points) Determine all the values for a and b for which the vector $\vec{b}$ is a unique linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Here we should not have any free variables and hence $a \neq 8$ and b anything.

I leave it up to you for diving out the points.

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VI: a) (7 points) Using the normal equations find the least square solution of the inconsistent system $A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \ , \ \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

This is pretty straight forward: The normal equations are $A^T A\vec{x} = A^T\vec{b}$

$$A^T A = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and

$$A^T\vec{b} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

so that

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

I leave it up to you to distribute the points.

b) (7 points) (7 points) The matrix B has the QR factorization $B = QR$ where

$$Q = \frac{1}{9}\begin{bmatrix} 1 & 8 \\ 4 & -4 \\ 8 & 1 \end{bmatrix} \ , \ R = \begin{bmatrix} 9 & 9 \\ 0 & 18 \end{bmatrix} \ , \ \vec{c} = \begin{bmatrix} 13 \\ 7 \\ 5 \end{bmatrix}$$

Solve the least square problem $B\vec{x} = \vec{c}$.

The relevant equation here is $R\vec{x} = Q^T\vec{c}$ This leads to

$$Q^T\vec{c} = \begin{bmatrix} 9 \\ 9 \end{bmatrix}$$

Hence

$$\vec{x} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Again, I leave it up to you how to distribute the points.

c) (6 points) Find the projection of $\vec{c}$ onto the column space of B.

There are two ways of solving this problem. One is to compute $QQ^T\vec{c}$ and the other is to compute $QR\vec{x}$ both, of course, yield the same result

$$\begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix}.$$

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VII: (8 points) a) Find a basis for the column space and a basis for the null space of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 5 \\ 3 & 1 & 2 & 12 \end{bmatrix}.$$

Row reduction leads to

$$\begin{bmatrix} 1 & 2 & 3 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first three columns are pivotal, the first three column vectors of the original matrix are basis vectors for the column space. (In fact, any three of them are.) Thus

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

are basis vectors for the column space.

Back substitution for solving the homogeneous equation leads to the basis vector for the nil space

$$\begin{bmatrix} -3 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

b) (7 points) What are $rank(A^T)$ and $dimNul(A^T)$?

The dimension of the null space of A^T is 1 This follows from the fact that the null space of A^T is the orthogonal complement of the column space of A. The sine A^T is 4×4 the rank of A^T is 3.

Concerning grading: If the student worked out the row reduction in the wrong way and did not oversimplify the problem he should get full credit for part b).

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VIII: a) (7 points) Find all the eigenvalues of the matrix

$$B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 1 & 4 \\ 1 & 0 & 4 \end{bmatrix} .$$

(You do not have to find the eigenvectors).

The characteristic polynomial is very simple:

$$(3 - \lambda)(1 - \lambda)(4 - \lambda)$$

b) (3 points) Is B diagonalizable?

Yes, since the eigenvalues are distinct.

c) (10 points) The matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

has the eigenvalues 7 and -2. Find an orthonormal basis of eigenvectors for A.

The eigenvalue 7 is degenerate and one has to apply Gram Schmidt. I got the basis

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{18}} \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

for the eigenspace associated with the eigenvalue 7. For the eigenvalue -2, the eigenvector is

$$\frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

Since the orthonormal basis for the eigenspace of the eigenvalue 7 is not unique one has to be careful when grading. The best way is to check that the vectors are perpendicular to each other and both are perpendicular to the eigenvector associated with the eigenvalue 2. Do not take too many points of if the eigenvectors are not normalized.

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IX: Consider the curve given by the equation

$$-17x^2 + 18xy + 7y^2 = 10 \ .$$

a) (3 points) Find the associated 2×2 matrix.

$$\begin{bmatrix} -17 & 9 \\ 9 & 7 \end{bmatrix}$$

This should not be any problem for grading.

b) (5 points) Find the eigenvalues and normalized eigenvectors

$$-20 \ , \ \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

and

$$10 \ , \ \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Give half credit for each. If the vectors are not normalized take maybe one point off. This will not influence the next two questions.

c) (2 points) What is the type of the curve, hyperbola or ellipse?

Hyperbola, since one eigenvalue is positive and the other negative.

Grading this should not be a problem.

d) (5 points) Sketch this curve below.

When grading this problem please pay attention to what the axis represent. In the $\vec{u}$ variables the equation is given by

$$-2u^2 + v^2 = 1$$

The u-axis corresponds to the the eigenvector associated with the eigenvalue -20 and the v-axis corresponds to the eigenvector associated with the eigenvalue 10. Hence the graph is a hyperbola that cuts the line passing through the origin in the direction of the vector $\frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ at the points 1 and -1.

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X: True or false? You do not have to give a reason. Each problem counts 3 points.

No partial credit is given for these problems

a) A 3×5 matrix has a null space whose dimension is greater or equal than 2.

True

b) A 5×3 matrix has a column space whose dimension is 2. Then the dimension of the null space of A is 1

True

c) The eigenvectors of a symmetric square matrix do necessarily form an orthonormal basis.

False

d) A 2×2 matrix satisfies the equation (as matrices) $A^2 = 4I$. Then, necessarily, A must have 2 and -2 as eigenvalues.

False

e) A 5×3 matrix has a column space whose dimension is 2. Then the dimension of the null space of A^T is 3

True