Practice Final Exam Solutions for Calculus II, Math 1502, December 5, 2013

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This test is to be taken without calculators and notes of any sorts. The allowed time is 2 hours and 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414 . . . . Show your work, otherwise credit cannot be given. **Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.**

Problem	Score
I	
II	
III	
IV	
V	
VI	
VII	
VIII	
IX	
X	
XI	
XII	
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Problems related to Block 1:

I: (15 points) Compute with an error less than 10^{-3}

$$\int_2^3 e^{\frac{1}{x^2}} dx \ .$$

$$e^y = \sum_{k=0}^{n} \frac{y^k}{k!} + \frac{e^c y^{n+1}}{(n+1)!}$$

where c is some number between 0 and y. Now we set $y = \frac{1}{x^2}$ and note that since x ranges between 2 and 3 the variable y ranges between $1/4$ and $1/9$. Hence we know that c can be a number that must be somewhere between 0 and $1/4$ and since e^y is monotone increasing we take $c = 1/4$ to obtain an upper bound on the remainder of the form

$$\frac{e^{1/4} y^{n+1}}{(n+1)!} \ .$$

Now from what we know about the exponential function $e < 3$ and hence $e^{1/4} < 3^{1/4}$ which is some number less than 2. Thus we find that

$$\begin{aligned} 0 &< \int_2^3 \left[e^{\frac{1}{x^2}} - \sum_{k=0}^n \frac{x^{-2k}}{k!} \right] dx \leq \frac{2}{(n+1)!} \int_2^3 x^{-2(n+1)} dx \\ &= \frac{2}{(n+1)!} \frac{1}{2n+1} \left[2^{-2n-1} - 3^{-2n-1} \right] < \frac{2}{(n+1)!} \frac{1}{2n+1} 2^{-2n-1} \ . \end{aligned}$$

If we choose $n = 3$ we find that the remainder of the integral is bounded by

$$\frac{2}{4!} \frac{1}{7} 2^{-7} = \frac{1}{84 \times 128} < \frac{1}{1000} \ .$$

Integrating the sum in the integral yields

$$\sum_{k=0}^{3} \left[2^{-2k+1} - 3^{-2k+1} \right] \frac{1}{k!(2k-1)} \ .$$

II: a) (7 points) Compute the limit

$$\lim_{x \to 0} \frac{e^x - \cos x - \sin x}{x^3}$$

Using the Taylor expansion for all the functions in the numerator yields

$$e^x - \cos x - \sin x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots - 1 + \frac{x^2}{2} + \cdots - x - \frac{x^3}{3!} + \ldots$$

The leading order coefficient is x^2 and hence the limit does not exist.

b) (8 points) Does the improper integral

$$\int_0^1 \frac{1}{x^2} e^{\frac{1}{x}} dx$$

exist? If yes, compute it.

We have to compute

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{1}{x^2} e^{\frac{1}{x}} dx$$

and using the substitution

$$u = \frac{1}{x}$$

this integral can be rewritten as

$$\int_1^{\frac{1}{\varepsilon}} e^u du = e^{\frac{1}{\varepsilon}} - e^{-1}.$$

Clearly the limit as $\varepsilon \to 0$ does not exist.

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Problems related to Block 2:

III: a) (7 points) Is the series

$$\sum_{k=0}^{\infty}(-1)^k \frac{(k!)^2}{k^{2k}}$$

convergent? Is it absolutely convergent?

Using the ratio test we get that

$$\frac{((k+1)!)^2}{(k+1)^{2k+2}} \frac{k^{2k}}{(k!)^2} = \left(\frac{k}{k+1}\right)^{2k}$$

which converges to $\frac{1}{e^2} < 1$. Hence the series is absolutely convergent and hence, in particular, convergent.

b) (8 points) Find the interval of convergence of the power series

$$\sum_{k=3}^{\infty}(-1)^k \frac{\ln k}{k}(x-2)^k$$

We are going to use the ratio test. Set

$$a_k = \frac{\ln k}{k}|x-2|^k$$

and note that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\ln(k+1)k}{(k+1)\ln k}|x-2| = |x-2|$$

hence the interval of convergence contains the interval $(1,3)$. Looking at the end point $x = 3$ we find the series

$$\sum_{k=3}^{\infty}(-1)^k \frac{\ln k}{k}$$

which converges since it is alternating and the coefficients decrease monotonically to zero. At $x = 1$, however, the series diverges, since it is given by

$$\sum_{k=3}^{\infty} \frac{\ln k}{k}$$

and diverges, by the limit comparison test with $1/k$.

IV: (15 points) Solve the initial value problem

$$y' - \frac{1}{x^2} y = e^{-\frac{1}{x}} \ , \ y(1) = \frac{2}{e} \ .$$

Multiply the equation by the integrating factor $e^{\frac{1}{x}}$

$$e^{\frac{1}{x}} y' - \frac{1}{x^2} e^{\frac{1}{x}} y = \left(e^{\frac{1}{x}} y \right)' = 1 \ .$$

Hence

$$y(x) = x e^{-\frac{1}{x}} + c e^{-\frac{1}{x}}$$

where c s a constant. $y(1) = \frac{2}{e}$ yields $c = 1$ and hence our solutions is

$$y(x) = (x+1) e^{-\frac{1}{x}} \ .$$

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Problems related to Block 3:

V: (15 points) Consider the system of equations

$$2x + y + z = b$$
$$x + y - 2z = 2$$
$$x - y + az = -1$$

Determine all values for a and b for which this system has a) non solution, b) exactly one solution, c) infinitely many solutions. In the case b) and c) Compute all the solutions in terms of a and b. The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 1 & | & b \\ 1 & 1 & -2 & | & 2 \\ 1 & -1 & a & | & -1 \end{bmatrix}$$

Switching the first and second row leads to

$$\begin{bmatrix} 1 & 1 & -2 & | & 2 \\ 2 & 1 & 1 & | & b \\ 1 & -1 & a & | & -1 \end{bmatrix}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 1 & -2 & | & 2 \\ 0 & -1 & 5 & | & b-4 \\ 0 & 0 & a-8 & | & 5-2b \end{bmatrix}$$

If $a = 8$ and $2b \neq 5$ there is no solution. If $a \neq 8$ there is always a unique solutions and if $a = 8$ and $2b = 5$ there are infinitely many solutions.

If $a \neq 8$ we can use back substitution and obtain:

$$z = \frac{5-2b}{a-8} \ , \ y = 5\frac{5-2b}{a-8} + 4 - b \ , \ x = -3\frac{5-2b}{a-8} + b - 2$$

If $a = 8$ and $2b = 5$ then the row reduced augmented matrix is

$$\begin{bmatrix} 1 & 1 & -2 & | & 2 \\ 0 & -1 & 5 & | & -\frac{3}{2} \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

and we find $z = t$, $y = 5t + \frac{3}{2}$ and $x = -3t + \frac{1}{2}$.

VI: (15 points) Consider the three vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}.$$

Determine all the values for a and b for which the vector

$$\vec{b} \begin{bmatrix} b \\ 2 \\ -1 \end{bmatrix}$$

is in the span of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Determine all the values for a and b for which the vector $\vec{b}$ is a unique linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

We have to row reduce the matrix

$$\begin{bmatrix} 2 & 1 & 1 & b \\ 1 & 1 & -2 & 2 \\ 1 & -1 & a & -1 \end{bmatrix}$$

which is row equivalent to

$$\begin{bmatrix} 1 & -1 & a & -1 \\ 0 & 2 & -2-a & 3 \\ 0 & 0 & 8-a & 2b-5 \end{bmatrix}$$

We have to check for which values of A, b the system is consistent. For all $a \neq 8$ the vector $\vec{b}$ is in the span of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ no matter what the value of b is. If $a = 8$ then b must be $5/2$ for the system to be consistent and hence again, $\vec{b}$ is in the span of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

In this last case there is a free variable and hence $\vec{b}$ is not a unique linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. If $a \neq 8$ and only in this case is $\vec{b}$ a unique linear combination of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. The first three columns are pivotal.

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VII: (15 points) Consider the matrix

$$A = \begin{bmatrix} 2 & 3 & 5 & 6 \\ 1 & 0 & 1 & 3 \\ 4 & 1 & 5 & 12 \\ 2 & 1 & 4 & 7 \end{bmatrix}$$

Find a basis for $Col(A)$ and for $Nul(A)$ as well as for $Col(A^T)$ and for $Nul(A^T)$. Try do this with a little computation as possible. Row reducing A yields

$$\begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix} \tag{0.1}$$

as a basis vector for the nullspace of A. Thus we can say that the dimension of $Col(A)$ equals 3. The row reduced matrix is

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and since the first three columns are pivotal, the first three column vectors of the matrix A form a basis for $Col(A)$.

Since the orthogonal complement of $Col(A^T)$ is the nullspace of A we have we have that the orthogonal complement of $Nul(A)$ is $Col(A^T)$. Thus, we have to find three linearly independent vectors that are perpendicular to the vector in (0.1). Thus we have to solve

$$-2w + x - y + z = 0$$

which leads to the one-one parametrization $z = t, y = s, x = r$ and $w = \frac{1}{2}[r - s + t]$. Hence we get

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

as a basis for $Col(A^T)$. To find a basis for $Nul(A^T)$ we remember that this space is the orthogonal complement of $Col(A)$. Hence, we have to find a vector perpendicular to the first three column vectors which can be found by row reducing the system

$$\begin{bmatrix} 2 & 1 & 4 & 2 \\ 3 & 0 & 1 & 1 \\ 5 & 1 & 5 & 4 \end{bmatrix}$$

which yields

$$\begin{bmatrix} 2 & 1 & 4 & 2 \\ 0 & 3 & 10 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and from which we get that the basis for the nullspace of A^T is

$$\begin{bmatrix} 1 \\ 10 \\ -3 \\ 0 \end{bmatrix}$$

VIII: (15 points) Diagonalize the matrices

$$a) \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix} .$$

The eigenvalue 6 has the eigenvector

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Perpendicular to this is the vector

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

which is an eigenvector with eigenvalue 3. Thus the vector

$$\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

which is perpendicular to both must be an eigenvector, since the matrix is symmetric. The associated eigenvalue is 3 also.

$$b) \quad \begin{bmatrix} 6 & 9 \\ 4 & 11 \end{bmatrix}$$

The characteristic polynomial is $\mu^2 - 17\mu + 30 = (\mu - 15)(\mu - 2)$. The iegenvector associated with $\mu_1 = 15$ is the vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the one associated to $\mu_2 = 2$ is

$$\begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

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IX: (20 points) Solve the recursive relation, i.e., find a_n for arbitrary values of n,

$$a_{n+1} = 8a_n + 9a_{n-1}$$

with $a_0 = a_1 = 1$.

Writing

$$\vec{x}_n = \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix}$$

we can write the recursion as

$$\vec{x}_{n+1} = A\vec{x}_n$$

where

$$A = \begin{bmatrix} 8 & 9 \\ 1 & 0 \end{bmatrix} \; .$$

The solution can be gotten via

$$\vec{x}_n = A^{n-1}\vec{x}_1 \; .$$

A is diagonalized by

$$A = \begin{bmatrix} 9 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix} \; .$$

Hence

$$A^{n-1} = \begin{bmatrix} 9 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 9^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & 1 \\ -1 & 9 \end{bmatrix}$$

Applying this to the initial condition $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ we find that

$$a_n = \frac{1}{5}\left(9^n + 4(-1)^n\right) \; .$$

X: (20 points) Find the least square solution for the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -1 & -4 \\ 1 & 3 & 0 \\ 1 & 3 & 6 \\ 1 & -1 & 2 \end{bmatrix}$$

and

$$\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} .$$

Solve the problem in two ways, once using the normal equations and then using the QR factorization.

The normal equation $A^T A\vec{x} = A^T\vec{b}$:

$$A^T A = \begin{bmatrix} 4 & 4 & 4 \\ 4 & 20 & 20 \\ 4 & 20 & 56 \end{bmatrix}$$

and

$$A^T\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

and the solution is

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

The QR factorization: The Gram-Schmidt procedure applied to the vectors in the column space leads to the matrix

$$Q = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

The R matrix is then given by

$$R = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 6 \end{bmatrix}$$

It remains to solve the equation $R\vec{x} = Q^T\vec{b}$. Since

$$Q^T\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

the solution is

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

as before.

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XI: (15 points) Graph the curve given by the equation

$$11x^2 - 6xy + 19y^2 = 10 \ .$$

The associated matrix is given by

$$\begin{bmatrix} 11 & -3 \\ -3 & 19 \end{bmatrix}$$

whose characteristic polynomial is $\mu^2 - 30\mu + 20 = (\mu - 10)(\mu - 20)$. Hence the eigenvalues are $\mu_1 = 10, \mu_2 = 20$. The associated eigenvectors are

$$\vec{u}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and

$$\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix} \ .$$

In the $u - v$ plane the cureve is given by

$$u^2 + 2v^2 = 1$$

which is an ellipse whose semiaxis in the u–direction has length 1 and whose semiaxis in the direction v has length $1/\sqrt{2}$. To get the picture in the $x - y$ plane we have to rotate the $u - v$ picture by the rotation matrix

$$U = [\vec{u}_1, \vec{u}_2] = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \ .$$

XII: (4 points each) Prove or find a counterexample:

a) Row operations do not change the column space of a matrix.

This is false. Take the matrix

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The column space is obviously one dimensional and spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ After row reduction the matrix is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

whose column space is spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

b) Row operations do not change the null space of a matrix

This is true. Since the row operations do not change the solutions set.

c) A 3×3 matrix has the eigenvalues $1, 2, 2$. Is it necessarily diagonalizable?

No, it is not. The matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

The eigenvalue 1 has the corresponding eigenvector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and for the eigenvalue 2 we get a single eigenvector (up to a non-zero multiple)

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

d) A real 3×3 matrix must always have at least one real eigenvalue.

This is true. The characteristic polynomial of a 3×3 matrix A is given by

$$p(\lambda) := \det(A - \lambda I) = -\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$

where a_1, a_2, a_3 are numbers. As λ tends to $+\infty$, $p(\lambda)$ tends to $-\infty$ and when λ tends to $-\infty$, $p(\lambda)$ tends to $+\infty$. By the intermediate value theorem $p(\lambda$ must have a real zero and hence A must have at least one real eigenvalue.

e) If a matrix A has the QR factorization $A = QR$ then the equation $R\vec{x} = Q^T\vec{b}$ has a solution for any $\vec{b}$.

This is true. If $A = QR$ then we know that QQ^T is the projection onto $Col(A)$. Hence the equation

$$QR\vec{x} = QQ^T\vec{b}$$

has always a solution. Multiplying this expression from the left by Q^T and using the $Q^TQ = I$ we find that

$$R\vec{x} = Q^T\vec{b}$$

must have a solution.