Test 1 for Calculus II, Math 1502 K1 - K6, September 11, 2013

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: Calculate the limits:
a) (8 points)

$$\lim_{x\to 0} \frac{\cos x^2 - 1}{x^4} \ .$$

Simplify first. With $y = x^2$ This is the same problem as

$$\lim_{y\to 0} \frac{\cos y - 1}{y^2} \ ,$$

which by l'Hôpital's rule

$$= \lim_{\to 0} \frac{-\sin(y)}{2y} = -\frac{1}{2} \ .$$

b) (8 points)

$$\lim_{x\to 0} \frac{x - \int_0^x e^{-t^2/3} dt}{x^3}$$

By l'Hôpital's rule this equals

$$\lim_{x\to 0} \frac{1 - e^{-x^2/3} dt}{3x^2} = \lim_{y\to 0} \frac{1 - e^{-y/3} dt}{3y} = \lim_{y\to 0} \frac{e^{-y/3} dt}{9} = \frac{1}{9}$$

c) (9 points)

$$\lim_{n\to\infty} n(\sqrt{n^2+1} - n)$$

Multiply by the conjugate

$$\lim_{n\to\infty} n(\sqrt{n^2+1} - n) = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1} + n)} = \frac{1}{2} \ .$$

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II:(25 points) a) Decide which of the following improper integrals exists and compute its values if it exists:
a) (8 points)

$$\int_0^\infty e^{-x} x dx \ ,$$

Integration by parts:

$$\int_0^L e^{-x} x dx = -e^{-x} x|_0^L + \int_0^L e^{-x} dx = Le^{-L} + 1 - e^{-L} \ ,$$

which converges as $L \to \infty$ to 1.
b) (8 points)

$$\int_2^\infty \frac{1}{x \ln(x)^2} dx \ ,$$

Substitution $u = \ln x$ yields

$$\int_2^L \frac{1}{x \ln(x)^2} dx = \int_{\ln 2}^{\ln L} \frac{1}{u^2} dx = \frac{1}{\ln 2} - \frac{1}{\ln L}$$

which converges to $\frac{1}{\ln 2}$ as $L \to \infty$.
c) (9 points)Does the following integral exist:

$$\int_0^\infty \frac{1}{\sqrt{x} + x^2} dx$$

Split the integral into

$$\int_0^1 \frac{1}{\sqrt{x} + x^2} dx + \int_1^\infty \frac{1}{\sqrt{x} + x^2} dx$$

The first integral exists by comparison since

$$\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{\sqrt{x}}$$

which is integrable. The second integral is also finite, again by comparison, since

$$\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{x^2}$$

and $\int_1^\infty \frac{1}{x^2} dx$ exists.

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III: a) (11 points) Solve the initial value problem

$$y' + \frac{1}{x}y = 1$$

with initial condition $y(1) = 1$.

The function

$$x$$

is an integrating factor and

$$(xy)' = x$$

and so

$$xy = \frac{x^2}{2} + C$$

The general solutions is

$$y(x) = \frac{x}{2} + \frac{C}{x} \ .$$

$y(1) = 1$ yields

$$y(x) = \frac{1}{2}(x + \frac{1}{x}) \ .$$

b) (14 points) On an arctic expedition food is stored outside at -20° C. At some time the food is brought in a room with temperature 20° C and after two hours the food has a temperature of 0°C. How How much does one have to wait until the food has a temperature of 10°C?

Newton's law of cooling says that

$$\frac{dH}{t} = -k(H - H_s)$$

where $H_s = 20$, the room temperature. The solutions is

$$H(t) = H_s + Ce^{-kt}$$

Now,

$$H(0) = -20$$

so that $C = -40$. Thus, our solution is

$$H(t) = 20 - 40e^{-kt}$$

Since

$$H(2) = 0$$

we have that

$$40e^{-2k} = 20$$

and $e^{-2k} = \frac{1}{2}$. Now

$$10 = H(T) = 20 - 40e^{-kT}$$

or

$$e^{-kT} = \frac{1}{4} = e^{-4k}$$

hence $T = 4$.

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IV: a) (7 points) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$$

Limit comparison test

$$\lim_{n\to\infty} \frac{n^2(n-2)}{n^3 - n^2 + 3} = 1$$

and hence the series is convergent since

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent. b) (7 points) Sum the series

$$\sum_{k=2}^{\infty} \frac{2^{k+3}}{3^k} .$$

$$\sum_{k=2}^{\infty} \frac{2^{k+3}}{3^k} = 2^3 \sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k = 2^3 \left(\frac{2}{3}\right)^2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = \frac{2^5}{3} = \frac{32}{3}$$

c) (11 points) Compute the limit

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k}}{\ln n}$$

Using the ideas for the proof of the integral test, we know that

$$\int_1^n \frac{1}{x} dx < \sum_{k=1}^{n} \frac{1}{k} < 1 + \int_1^n \frac{1}{x} dx$$

so that

$$\ln n < \sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n$$

Therefore

$$1 < \frac{\sum_{k=1}^{n} \frac{1}{k}}{\ln n} < 1 + \frac{1}{\ln n}$$

Hence, as $n \to \infty$ we get that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \frac{1}{k}}{\ln n} = 1$$