Practice Test 2B for Calculus II, Math 1502, September 30, 2013

PRINT Name:

PRINT Section:

PRINT Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414 .... Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: (25 points) a) Does the series

$$\sum_{k=2}^{\infty}(-1)^k \frac{1}{k \ln k}$$

converge absolutely?

The series

$$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$$

does not converge since the integral

$$\int_2^{\infty} \frac{1}{x \ln x} dx$$

is divergent.

b) Consider the series

$$\sum_{k=0}^{\infty}(-1)^k \frac{2^k}{(2k)!} \ .$$

Does this series converge? If yes, calculate the first five digits after the decimal point of this limit.

The series is an alternating series, $u_k = \frac{2^k}{(2k)!}$ is decreasing and converges to zero. If L denotes the limit we have that

$$|L - s_n| < u_{N+1} = \frac{2^{N+1}}{(2N+2)!} \leq$$

For $N = 5$ we get that the error is less than 1.4×10^{-7}. For $N = 4$ we have that the error is 9×10^{-6} which might influence the fifth digit. The answer is

$$\sum_{k=0}^{5}(-1)^k \frac{2^k}{(2k)!} \ .$$

c) Let

$$\sum_k a_k x^k$$

be a power series and assume that it converges at $c > 0$. True or false:

1) The series necessarily converges for all $x < c$. NO
2) The radius of convergence R satisfies necessarily $R \geq c$. YES
3) The radius of convergence R satisfies necessarily $R \leq c$. NO
4) The series converges absolutely for x with $|x| < c$. YES

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II: (25 points) a) Write the power series expansion at $x = 0$ of the function

$$\int_0^x \frac{1}{1+t^4} dt \ .$$

For which values of x does the series converge? What is the radius of convergence?

$$\frac{1}{1+t^4} = \sum_{k=0}^{\infty} (-1)^k t^{4k}$$

so that

$$\int_0^x \frac{1}{1+t^4} dt = \sum_{k=0}^{\infty} (-1)^k \frac{x^{4k+1}}{4k+1}$$

The radius of convergence $R = 1$. It converges absolutely for all $|x| < 1$. It converges at $x = 1$ and diverges at $x = -1$.

b) What is the radius of convergence of the power series

$$\sum_{k=1}^{\infty} (1 + \frac{1}{k})^{k^2} x^k \ ?$$

Using the root test, we have to compute

$$\lim_{k \to \infty} (1 + \frac{1}{k})^{1/k} |x| = e|x|$$

and hence the radius of convergence is $1/e$.

c) Find the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{1}{k}(x-2)^k 2^{-k}$$

The series converges absolutely for all x with $|x-2| < 2$, that is for

$$0 < x < 4 \ .$$

At $x = 4$ the series diverges, because it is the harmonic series, and at $x = 0$ it converges since it is the alternating harmonic series.

$$\sum_{k=2}^{\infty} \frac{\log k}{k^2} x^k$$

The radius of convergence is 1, i.e., the series converges for all x with $|x| < 1$. At $x = 1$ the series reduces to

$$\sum_{k=2}^{\infty} \frac{\log k}{k^2}$$

which can be analyzed using the limit comparison test. Indeed,

$$\lim k \to \infty \frac{\log k}{k^2} k^{3/2} = 0$$

and since

$$\sum_{k=2}^{\infty} \frac{1}{k^{3/2}}$$

converges so does the series

$$\sum_{k=2}^{\infty} \frac{\log k}{k^2} \ .$$

This argument also shows that the series converges absolutely at $x = -1$.

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III: (25 points) a) Find the intersection the line

$$\vec{x}(t) = \langle 1, 2, 1\rangle + t\langle 0, 4, 2\rangle$$

with the plane

$$x + y + z = 3$$

The general point on the line is $x = 1, y = 2 + 4t, z = 1 + 2t$. In order that this point is on the plane we need that $1 + 2 + 4t + 1 + 2t = 3$ which means that $t = -\frac{1}{6}$. Hence the point of intersection is

$$x = 1, y = 4/3, z = 2/3 \ .$$

b) Find the angle between the planes

$$2x - y + 3z = 2 \ , \ \ 5x + 5y - z = 4$$

The vectors normal to the planes are $\langle 2, -1, 3\rangle$ and $\langle 5, 5, -1\rangle$. The angle between these vectors is

$$\frac{2}{\sqrt{14}\sqrt{51}}$$

so that

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{14}\sqrt{51}}\right)$$

c) Find the line that forms the intersection of the two planes

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IV: (25 points) a) Find the distance of the tip of the vector $\langle 2, -1, 3\rangle$ to the plane

$$2x + 4y - z = -1$$

Pick any point in the plane, e.g., the tip of the vector $\langle 0, 0, 1\rangle$ is one this plane. The vector starting at this tip and ending at the tip of the vector $\langle 2, -1, 3\rangle$ is given by $\langle 2, -1, 3\rangle - \langle 0, 0, 1\rangle = \langle 2, -1, 2\rangle = \vec{u}$. Now, we project this vector $\vec{u}$ onto the vector normal to the plane $\vec{a} = \langle 2, 4, -1\rangle$ and get

$$P_{\vec{a}}\vec{u} = \frac{\vec{u} \cdot \vec{a}}{|\vec{a}|^2}\vec{a}$$

The length of this vector is the distance, i.e. $|\frac{\vec{u}\cdot\vec{a}}{|\vec{a}|}| = \frac{2}{\sqrt{21}}$

b) Find the distance of the tip of the vector $\langle 1, 2, 3\rangle$ to the line $\vec{x}(t) = \langle 1, 0, 2\rangle + t\langle 1, -2, 3\rangle$.

Here the thinking is different from the problem about the plane. Pick any point on the line, e.g., $\langle 1, 0, 2\rangle$ and consider the vector

$$\vec{p} = \langle 1, 2, 3\rangle - \langle 1, 0, 2\rangle = \langle 0, 2, 1\rangle \ .$$

The distance we are trying to compute is given by $|\vec{p}| \sin\theta$ where θ is the angel between the direction vector $\vec{v} = \langle 1, -2, 3\rangle$ of the line and the vector $\vec{p}$. We know that

$$|\vec{p} \times \vec{v}| = |\vec{p}||\vec{v}| \sin\theta$$

and hence the distance is given by

$$\frac{|\vec{p} \times \vec{v}|}{|\vec{v}|} = \frac{|\langle 8, 1, -2\rangle|}{\sqrt{14}} = \frac{\sqrt{69}}{\sqrt{14}} \ .$$