Practice Test 4B for Calculus II, Math 1502, November 9, 2013

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PRINT Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

PRINT your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: a) (10 points) Let S_1 and S_2 be two subspaces of $\mathcal{R}^n$. Show that the intersection of these subspaces is again a subspace.

Suppose $\vec{v}_1$ and $\vec{v}_2$ are vectors both in S_1 and in S_2. Then $\vec{v}_1 + \vec{v}_2$ is in S_1 since S_1 is a subspace. Moreover, $\vec{v}_1 + \vec{v}_2$ is in S_2 since S_2 is a subspace. Hence $\vec{v}_1 + \vec{v}_2$ is in both and hence in the intersection. Likewise, suppose $\vec{u}$ is a vector in the intersection of S_1 and S_2 and c a scalar. Then $c\vec{u}$ is in S_1 since S_1 is a subspace. Of course it is also in S_2 since S_2 is a subspace. Thus $c\vec{u}$ is in both and hence in the intersection. Thus, the intersection of S_1 and S_2 is a subspace. Note, the union of the two subspaces is not a subspace (why?).

b) (10 points) Consider the subspacesS_1 given by the span of the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \ , \ \vec{v}_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

and S_2 which is given by the span of the vectors

$$\vec{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \ , \ \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \ .$$

Find the intersection of S_1 and S_2.

We have to find all vectors that are in the span of $\vec{v}_1, \vec{v}_2$ as well as in the span of $\vec{w}_1, \vec{w}_2$, i.e., we have to solve the system of equation given by

$$s\vec{v}_1 + t\vec{v}_2 = x\vec{w}_1 + y\vec{w}_2 \ .$$

The solutions of this system we can get by considering the augmented matrix

$$\begin{bmatrix} 1 & 3 & -2 & -1 \\ 2 & 2 & -1 & -1 \\ 3 & 1 & 0 & -2 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 3 & -2 & -1 \\ 0 & -4 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and backward substitution yields that

$$\begin{bmatrix} s \\ t \\ x \\ y \end{bmatrix} = x \begin{bmatrix} -1 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

Thus, we have that

$$-x\vec{v}_1 + x3\vec{v}_2 = 4x\vec{w}_1$$

where x is any number. Hence, the intersection is any multiple of the vector $\vec{w}_1$, i.e.,

$$x \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ x \text{ in } \mathcal{R}$$

and hence a line passing through the origin.

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II: a) (10 points) By computing the determinant of a matrix decide for which values of a the vectors

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} , \begin{bmatrix} 7 \\ a \\ 1 \end{bmatrix} , \begin{bmatrix} -10 \\ 6 \\ 6 \end{bmatrix}$$

are linearly dependent.

The determinant is given by $26(a-3)$ and hence vanishes precisely for $a = 3$. Hence only for $a = 3$ are the vectors linearly dependent.

b) (10 points) By making as few computations as possible compute the determinant of the matrix

$$\begin{bmatrix} 5 & 6 & 6 & 6 & 6 \\ 3 & 4 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 4 & 5 & 5 & 5 & 4 \end{bmatrix}$$

First we swap row one and row three. In the resulting matrix we swap row two and row four. In this new matrix we swap row three and row four and get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 3 & 4 & 4 & 0 & 0 \\ 5 & 6 & 6 & 6 & 6 \\ 4 & 5 & 5 & 5 & 4 \end{bmatrix}$$

The determinant of this matrix is

$$1 \cdot 3 \cdot 4 \cdot \det \begin{bmatrix} 6 & 6 \\ 5 & 4 \end{bmatrix} = -72 \ .$$

Since we had to use an odd number of row swaps the determinant is 72.

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III: a) (10 points) Consider the matrices

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the inverse of $E_1 \cdot E_2$.

The inverse matrices of E_1 and E_2 are

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \ , \ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence

$$(E_1 E_2)^{-1} = E_2^{-1} E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

b) (10 points) Given

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Find all the vectors $\vec{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ so that the equation

$$LU\vec{x} = \vec{b}$$

has a solution.

We have to solve $L\vec{y} = \vec{b}$ and then $U\vec{x} = \vec{y}$. In order that there exists a solution we must have that the system $U\vec{x} = \vec{y}$ is consistent, i.e., the third component of $\vec{y}$ must vanish. Setting

$$\vec{y} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

we have that $u = a, v = 2a - b, w = c - 4a + 2b$. Hence the set of vectors $\vec{b}$ for which the equation $LU\vec{x} = \vec{b}$ has a solution are all those that satisfy the equation $c - 4a + 2b = 0$.

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IV: (20 points) Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

(Hint: One eigenvector is easy to guess).

Obviously

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue 6. Now we calculate the characteristic polynomial

$$\det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & 2-\lambda & 2 \\ 3 & 2 & 1-\lambda \end{bmatrix} = -\lambda(\lambda^2 - 4\lambda - 12) = -\lambda(\lambda - 6)(\lambda + 2) \ .$$

hence the eigenvlaues are $0, 6, -2$. The eigenvectors in the same order of appearance are

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \ .$$

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V: (5 points) each. Prove or find a counterexample. Let A be an $n \times m$ matrix.

a) $Nul(A)$ is a subspace of $\mathcal{R}^n$.

False. $Nul(A)$ is a subspace of $\mathcal{R}^m$. $Col(A)$ is a subspace of $\mathcal{R}^n$.

b) $\dim Col(A)$ is less than the smaller of the numbers n and m.

True. Since $Col(A)$ is a subspace of $\mathcal{R}^n$ we have that $\dim Col(A) \leq n$. Since there are m columns in the matrix A, the dimension of $\dim Col(A)$ cannot be bigger than m and hence $\dim Col(A) \leq m$.

c) If the dimension of $Col(A)$ is 1, then the dimension of $Nul(A)$ is $m-1$.
True. We have that

$$\dim Col(A) + \dim Nul(A) = m \text{ or } \dim Nul(A) = m - \dim Col(A) \ .$$

d) If λ is an eigenvalue of A then $\dim Nul(A - \lambda I) = 1$.

This is false. Just take A to be the 2×2 identity matrix. Here $\dim Nul(A - 2I) = 2$.