# 1. Quiz 4 Solutions

**Problem 1:**

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \end{bmatrix}$$

a) Find a basis for the column space of $A$. Row reduction leads to

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and third are pivot columns and hence

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

form a basis for the column space.

b) Find a basis for the null space of $A$. The second and fourth variable are free and hence $x_3 = -\frac{3}{2}x_4$ and $x_1 = -2x_2 - 3x_3 - 4x_4 = -2x_2 + \frac{9}{2}x_4 - 4x_4 = -2x_2 + \frac{1}{2}x_4$. Hence any solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

and the two vectors are a basis for the null space.

**Problem 2:** a)

$$A \begin{bmatrix} 2 & 1 & 4 \\ 3 & 4 & 2 \\ 5 & 5 & 6 \end{bmatrix}, \vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

$$A\vec{u} = \begin{bmatrix} 9 \\ 12 \\ 21 \end{bmatrix}, A\vec{v} = \begin{bmatrix} 11 \\ 11 \\ 22 \end{bmatrix}.$$

Hence, $\vec{u}$ is not and eigenvector and $\vec{v}$ is an eigenvector with eigenvalue 11.

b) Find the volume of the parallelepiped determined by

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Have to compute the

$$\det \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= 1\det \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} - 2\det \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} + 2\det \begin{bmatrix} 2 & 1 \\ 2 & -2 \end{bmatrix}$$

which equals $-27$. Hence the volume is 27 units.

**Problem 3:** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Find the eigenvalues of $A$. For each eigenvalue find a basis for the corresponding eigenspace.

1 is clearly an eigenvalue and $\vec{e}_1$ is a corresponding eigenvector.

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 3-\lambda \end{bmatrix} = (1-\lambda)[\lambda^2 - 4\lambda + 4] = (1-\lambda)(\lambda - 2)^2$$

Hence 2 is a double eigenvalue. Row reducing

$$\begin{bmatrix} 1-2 & 2 & 3 \\ 0 & 1-2 & 1 \\ 0 & -1 & 3-2 \end{bmatrix}$$

yields

$$\begin{bmatrix} -1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and $x_3$ is the only free variable and the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

hence the vector $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$ yields a basis for the eigenspace associated with the eigenvalue 2. The vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is a basis for the eigenspace associated with the eigenvalue 1.

**Extra credit:** Find a $2 \times 2$ matrix such that the column space is the line $x_1 + 2x_2 = 0$ and the null space is the line $x_1 = x_2$.

The vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis for the column space and the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for the null space of the matrix. The first condition says that the matrix has to be of the form

$$\begin{bmatrix} -2a & -2b \\ a & b \end{bmatrix},$$

where $a, b$ should not both be zero, and the second condition requires that $a + b = 0$. Hence the set of matrices fulfilling the requirements are of the form

$$a \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

where $a \neq 0$.