Solutions for Practice Test 1A for Calculus II, Math 1502, September 5, 2013

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.
Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

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I: (25 points) Calculate the limits:
a)

$$\lim_{x\to 0} \frac{f(x)}{f^{-1}(x)} \ ,$$

where $f(x)$ is a differentiable and invertible function with $f(0) = 0$ and $f'(0) = 4$.

$$\frac{d}{dy} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

and by l'Hôpital's rule

$$\lim_{x\to 0} \frac{f(x)}{f^{-1}(x)} = \lim_{x\to 0} f'(x) f'(f(x)) = f'(0) f'(f(0)) = f'(0)^2 = 16 \ .$$

b)

$$\lim_{x\to 0} \frac{x - \int_0^x [\cos(t)]^2 dt}{x^3}$$

$$= \lim_{x\to 0} \frac{1 - \cos(x)^2}{2x^2} = \lim_{x\to 0} = \frac{1}{3}$$

c)

$$\lim_{x\to 0} \left(\frac{1}{\sin(2x)} - \frac{1}{\tan(2x)} \right) = \lim_{x\to 0} \frac{1 - \cos(2x)}{\sin(2x)} = 0$$

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II:(25 points) a) Decide which of the following improper integrals exists and compute its values if it exists:

$$a) \quad \int_0^\infty e^{-x}\cos(x)dx \ , \qquad b) \quad \int_0^\infty \frac{x}{1+x^2}dx$$

a) Integrate by parts twice

$$\int_0^b e^{-x}\cos(x)dx = -e^{-x}\cos(x)|_0^b - \int_0^b e^{-x}\sin(x)dx$$

$$= -e^{-x}\cos(x)|_0^b + e^{-x}\sin(x)|_0^b - \int_0^b e^{-x}\cos x dx$$

or

$$\int_0^b e^{-x}\cos(x)dx = \frac{1}{2}\left[1 - e^{-b}\cos(b) + e^{-b}\sin(b)\right]$$

which converges to $1/2$ as $b \to \infty$.
b)

$$\int_0^b \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2)|_0^b = \frac{1}{2}\ln(1+b^2)$$

which diverges as $b \to \infty$.
Use the comparison test to decide which of the following integrals exists:

$$c) \quad \int_0^\infty \frac{1}{[\sin(x)]^2 + x^2}dx \ , \quad d) \int_0^\infty \frac{x^2}{\sqrt{1+x^6}}dx$$

c) Since $|\sin x| \le |x|$ we have that

$$\frac{1}{[\sin(x)]^2 + x^2} \ge \frac{1}{2x^2}$$

which is not integrable near $x = 0$.
d) Split the integral into

$$\int_0^1 \frac{x^2}{\sqrt{1+x^6}}dx + \int_1^\infty \frac{x^2}{\sqrt{1+x^6}}dx$$

and note that the first is harmless while for the second we know that

$$\lim_{x\to\infty} \frac{\frac{x^2}{\sqrt{1+x^6}}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+x^{-6}}}x = 1$$

But

$$\frac{1}{x}$$

is not integrable and hence the original function is not integrable by the limit comparison test.

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III: (25 points) Which of the following series is convergent or divergent. Reason carefully!
a)

$$\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^{k^2}$$

Note that

$$\frac{k+1}{k} \ge 1$$

and hence the sum does not converge.
b)

$$\sum_{k=0}^{\infty} \frac{1}{(k+2)(k+3)} \ .$$

Using that

$$\frac{1}{(k+2)(k+3)} = \frac{1}{(k+2)} - \frac{1}{(k+3)}$$

the partial sums converge since it is a telescoping sum. The limit is $\frac{1}{2}$
c) Consider the convergent series

$$L = \sum_{k=0}^{\infty} \frac{1}{3^k}$$

Find the smallest n so that $0 < L - s_n < 10^{-3}$.
Note that

$$L - s_n = \sum_{k=n+1}^{\infty} \frac{1}{3^k} = \frac{1}{3^{n+1}} \sum_{k=0}^{\infty} \frac{1}{3^k} = \frac{1}{2 \cdot 3^n}$$

Now

$$2 \cdot 3^5 = 486 < 1000 \ , \ 2 \cdot 3^6 = 1458 > 1000$$

Hence $n = 6$ is the answer.

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IV: a) Solve the differential equation

$$\frac{dx}{dt} = \sin x$$

with initial condition $x(0) = \pi/2$.

Use separation of variables. You can check that

$$\frac{d}{dx}\ln(\tan(\frac{x}{2})) = \frac{1}{2}\frac{1}{\tan(\frac{x}{2})}\frac{1}{\cos(\frac{x}{2})^2} = \frac{1}{2}\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})}\frac{1}{\cos(\frac{x}{2})^2} = \frac{1}{2\sin(\frac{x}{2})\cos(\frac{x}{2})}$$

$$= \frac{1}{\sin x}$$

Hence

$$t = \int_0^t \frac{\dot{x}}{\sin x}dt = \ln(\tan(\frac{x(t)}{2})) - \ln(\tan(\frac{x(0)}{2}))$$

$$\ln(\tan(\frac{x(0)}{2})) = \ln(\tan(\frac{\pi}{4})) = \ln 1 = 0$$

and hence

$$x(t) = 2\tan^{-1}(e^t)$$

b) At a certain moment, a tank contains 100 liters of brine with a concentration 40 grams of salt per liter. The brine is continuously drawn off at a rate of 10 liters per minute and replaced by brine containing 20 grams salt per liter. Find the amount of salt in the tank at time t later.

Denote by $P(t)$ the amount of salt in the tank (measured in g). $P(0) =$ 4000 g.

$$\frac{dP}{dt} = \text{rate in} - \text{rate out}$$

$$\text{rate in} = 10 \times 20$$

$$\text{rate out} = \frac{P(t)}{100} \times 10$$

This leads to

$$\frac{dP}{dt} = 200 - \frac{P(t)}{10}$$

The solution is using the integrating factor $e^{t/10}$

$$P(t) = 2000 + Ce^{-t/10}$$

$$4000 = 2000 + C$$

and so $C = 2000$. Hence

$$P(t) = 2000(1 + e^{-t/10})\ .$$