Solution for Practice Test 1B for Calculus II, Math 1502, September 9, 2013

Name:

Section:

Name of TA:

This test is to be taken without calculators and notes of any sorts. The allowed time is 50 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$. Show your work, otherwise credit cannot be given.

Write your name, your section number as well as the name of your TA on EVERY PAGE of this test. This is very important.

Name:

Section:

Name of TA:

I: (25 points) a) Consider the recursive sequence $a_{n+1} = \sqrt{2+a_n}, n = 0, 1, 2\ldots$ and $a_0 = 0$. Assuming that the sequence converges, compute its limit.

Denote by $A = \lim_{n\to\infty} a_n$ which exists by assumption. Since the root is a continuous function we may interchange limit and root and get

$$A = \sqrt{2+A}$$

and hence

$$A^2 = 2 + A$$

This is a quadratic equation which can be readily solved and has the roots $2, -1$. Since all the $a_n \geq 0$ the limit must be positive and hence $A = 2$.

b) Compute the limit $\lim_{n\to\infty} a_n$ where

$$a_n = \frac{1}{\sqrt{n^2-1} - \sqrt{n^2+n}} \ .$$

Multiplying by the conjugate yields

$$\begin{aligned} a_n &= \frac{\sqrt{n^2-1} + \sqrt{n^2+n}}{[\sqrt{n^2-1} - \sqrt{n^2+n}][\sqrt{n^2-1} + \sqrt{n^2+n}]} \\ &= \frac{\sqrt{n^2-1} + \sqrt{n^2+n}}{[n^2-1-n^2-n]} = -\frac{\sqrt{n^2-1} + \sqrt{n^2+n}}{n+1} \end{aligned}$$

which converges to -2 as $n \to \infty$.

c) Express the number $0.\overline{123} = 0.123123\cdots$ as a ratio of two integers. Note that $0.\overline{123}$ can be written as

$$\begin{aligned} 1\left[\frac{1}{10} + \frac{1}{10^4} + \frac{1}{10^7} + \cdots\right] &+ 2\left[\frac{1}{10^2} + \frac{1}{10^5} + \frac{1}{10^8} + \cdots\right] \\ &+3\left[\frac{1}{10^3} + \frac{1}{10^6} + \frac{1}{10^9} + \cdots\right] \\ &= \left[\frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3}\right]\frac{1}{1-10^{-3}} \\ &= \frac{123}{999} = \frac{41}{333} \ . \end{aligned}$$

Name:

Section:

Name of TA:

II: (25 points) a) For what a does the limit

$$\lim_{x\to 0}\frac{\cos(x^2)-1}{x^a}$$

exist and is not zero?

$$a = 4$$

Use any test to decide which of the following integrals exists:

$$a)\quad \int_0^\infty \frac{1}{x+(x-1)^2}dx\ , \qquad b)\quad \int_{1/2}^{3/2}\frac{1}{x(\ln x)^2}dx$$

a) Note that

$$x+(x-1)^2 \geq \frac{3}{4}$$

and hence we only have to check convergence at ∞. Now do use $\frac{1}{x^2}$ as a comparison function and note that

$$\frac{\frac{1}{x+(x-1)^2}}{\frac{1}{x^2}} \to 1$$

as $x\to\infty$. Since

$$\int_1^\infty \frac{1}{x^2}dx$$

exists so does our integral.

As for b) not that the problem is at $x = 1$ since the logarithm vanishes there. For $h > 0$ but small

$$\int_{1/2}^{1-h}\frac{1}{x(\ln x)^2}dx = -\frac{1}{\ln(1-h)} - \frac{1}{\ln 2}$$

which diverges to $+\infty$ as $h \to 0$. Hence the integral does not exist. Likewise, one can analyze the other part too:

$$\int_{1+h}^{3/2} \frac{1}{x(\ln x)^2}\,dx = \frac{1}{\ln(1+h)} - \frac{1}{\ln(3/2)}$$

which also diverges to $+\infty$.

Name:

Section:

Name of TA:

III: (25 points) a) Solve the initial value problem

$$y' + 3x^2 y = x^2 \quad y(1) = 2$$

Integrating factor is

$$e^{x^3}$$

Hence

$$(e^{x^3} y)' = x^2 e^{x^3}$$

and

$$e^{x^3} y = \frac{1}{3} e^{x^3} + C$$

and the general solution is

$$y(x) = \frac{1}{3} + Ce^{-x^3}$$

$$2 = y(1) = \frac{1}{3} + \frac{C}{e}$$

So

$$C = \frac{5}{3} e$$

and

$$y(x) = \frac{1}{3} \left[1 + 5e^{-x^3+1} \right]$$

b) (from Thomas) An aluminum beam was brought in from the outside cold into a machine shop where the temperature was held at 65° F. After 10 minutes, the beam warmed to 35° F and after another 10 minutes to 50° F. Use Newton's law of cooling to compute the initial temperature of the beam.

Newton's law of cooling says that

$$\frac{dH}{dt} = -k(H - H_s)$$

where H_s is the temperature of the surrounding environment. The solution is

$$H(t) = Ce^{-kt} + H_s$$

We know that $H_s = 65$, $H(10) = 35$ and $H(20) = 50$ Fahrenheit. Hence we have the equations

$$-30 = Ce^{-10k} \ , \ -15 = Ce^{-20k} \ .$$

We are interested in C since $H(0) = H_s + C$. So

$$-\frac{30^2}{15} = \frac{C^2 e^{-20k}}{Ce^{-20k}} = C$$

and hence $C = -60$ and $H(0) = 5$ Fahrenheit.

Name:

Section:

Name of TA:

IV: (25 points)

a) Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a}$$

where $a > 0$. For which values of a is this series convergent and for which ones divergent.

Use the integral test which says that the series converges if and only if the integral

$$\int_2^{\infty} \frac{1}{x(\ln x)^a} dx$$

exists. But with $u = \ln x$,

$$\int_2^{L} \frac{1}{x(\ln x)^a} dx = \int_{\ln 2}^{\ln L} \frac{1}{u^a} du$$

The limit as $L \to \infty$ exists if $a > 1$ and if $a \leq 1$ it does not exist. Hence the series converges if $a > 1$ and diverges if $a \leq 1$.

b) Does the series

$$\sum_{k=0}^{\infty} \sqrt{\frac{n+1}{n^3+2}} \, ,$$

converge?

Use the limit comparison test with the series $\sum \frac{1}{n}$ which does not exist. Since

$$\lim_{n\to\infty} n\sqrt{\frac{n+1}{n^3+2}} = 1$$

our series diverges.

c) Find n so that the partial sum $s_n = \sum_{k=1}^{n} \frac{1}{k^4}$ estimates the value of the series $\sum_{k=1}^{\infty} \frac{1}{k^4}$ with an error of at most 10^{-6}.

We know that the series converges by the integral test. To refine the analysis consider

$$\sum_{k=N+1}^{\infty} \frac{1}{k^4}$$

We know that

$$\int_{N+1}^{\infty} \frac{1}{x^4} dx \leq \sum_{k=N+1}^{\infty} \frac{1}{k^4} \leq \int_{N}^{\infty} \frac{1}{x^4} dx$$

Draw two pictures!!!! Hence, computing the integrals yields

$$\frac{1}{3} \frac{1}{(N+1)^3} < \sum_{k=N=1}^{\infty} \frac{1}{k^4} < \frac{1}{3} \frac{1}{(N)^3}$$

If we denote by L the limit of the sum we have that the N-th partial sum s_N satisfies

$$L - s_N = \sum_{k=N}^{\infty} \frac{1}{k^4}$$

and hence

$$\frac{1}{3} \frac{1}{(N+1)^3} < L - s_N < \frac{1}{3} \frac{1}{(N)^3} \ .$$

The left side is not so interesting since we know that $L - s_N > 0$. Thus, if we choose $N = 100$ we know that

$$0 < L - s_N < \frac{1}{3} 10^{-6} \ .$$